

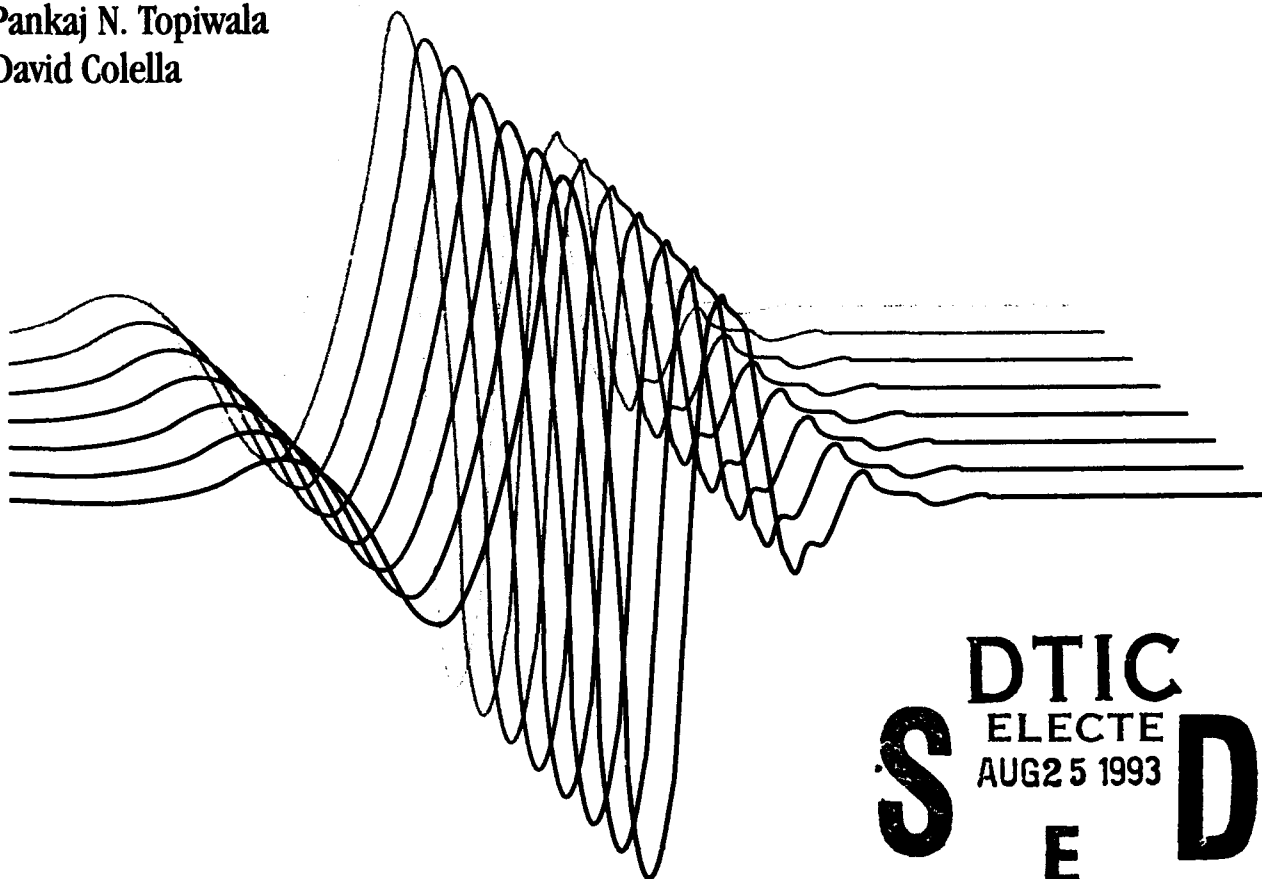
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## Introduction to Wavelets

M 93B0000076  
July 1993

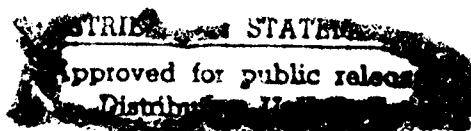
Pankaj N. Topiwala  
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Bedford, Massachusetts

# **Introduction to Wavelets**

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David Colella**

**The MITRE Corporation  
Burlington Rd.  
Bedford, MA 01730**

**15 March 1993**

## **Title Slide: Introduction to Wavelets**

This half-day course on wavelets is intended to be a general introduction to this exciting new topic in signal processing. Wavelets have been proven effective for a number of signal processing tasks, including compression and edge and transient detection. The computational complexity of a wavelet decomposition is  $O(N)$ , versus  $O(N \log N)$  for the FFT, making it very competitive. In addition, it has several specific advantages over the FFT in many applications. Two of these applications will be discussed in this course: transient detection and image compression.

Wavelets have been applied for a wide variety of tasks, and MITRE has been involved in wavelet research for a number of years, leading to over thirty publications. This course was developed in conjunction with W090 and W096 staff (who ran a successful 3-day wavelet course in Washington), and taught in Bedford by P. Topiwala. This introductory course will be followed by a second course covering the wavelet research and applications currently being developed at MITRE.

## **Course Outline**

- **Preliminaries**
- **Theory**
- **Applications**

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### **Course Outline**

**This course will be divided into an introduction to the theory and to the applications. Because of the wide audience of this course, a general summary of preliminary material is provided as well.**

## **Preliminaries**

- **Linear Algebra**
- **Linear Analysis**
- **Fourier Transform**

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### **Preliminaries**

The preliminaries include linear algebra in finite dimensions, linear analysis in infinite dimensions, and some well-known facts about the Fourier transform.

# Linear Algebra 1

- Vector space  $V$  over  $\mathbb{R}, \mathbb{C}$   $v, w \in V, \alpha, \beta \in \mathbb{R}, \mathbb{C} \Rightarrow \alpha v + \beta w \in V$
- Finite dimensions :  $\mathbb{R}^n, \mathbb{C}^n$
- Inner product :  $\langle v, w \rangle = \sum_1^n v_i \bar{w}_i$
- Norm :  $|v| = \sqrt{\sum_1^n v_i^2}$
- Basis  $\{e_i\}_1^n : v = \sum_1^n \alpha_i e_i, \alpha_i \in \mathbb{R}, \mathbb{C}$
- Orthonormal (ON) Basis :  $\alpha_i = \langle v, e_i \rangle$   
That is ,  $v = \sum_1^n \langle v, e_i \rangle e_i$
- Example :  $e_k = (0, \dots, 1, \dots, 0)$  1 in the kth spot (standard basis)

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## Linear Algebra 1

This slide covers the basic notions of a vector space and an orthonormal basis, including the definition of an inner product, and the standard orthonormal basis.

## Linear Algebra 2

- Linear Map :  $A: V \rightarrow W$   
 $A(\alpha v_1 + \beta v_2) = \alpha A(v_1) + \beta A(v_2)$
- Dual Basis :  $V, \{e_k\}$     $e_k^*: V \rightarrow \mathbb{R}, \mathbb{C}$   
 $e_k^*(e_l) = \delta_{kl} = \begin{cases} 1, & k = l \\ 0, & \text{else} \end{cases}$
- Matrix Rep .  $(V, \{e_k\}), (W, \{f_l\})$   
 $A(e_k) = \sum_l a_{kl} f_l$
- Example (standard)    $e_k^* = e_k^T = (0, \dots, 1, \dots, 0)$ , 1 in the kth spot  
 $A \leftrightarrow (a_{ij})$  If  $A(e_i) = \sum_{j=1}^n a_{ij} e_j$

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### Linear Algebra 2

A linear map between two vector spaces can be represented as a matrix, given bases for the vector spaces.

## Linear Algebra 3

- $V \oplus W$  has basis  $e_1, \dots, e_n, f_1, \dots, f_m$ ;  $\dim = n + m$
- $V \otimes W$  has basis  $\{e_i \otimes f_j\}$ ,  $i = 1, \dots, n; j = 1, \dots, m$ .  $\dim = n m$   
space of matrices of size  $n \times m$ , basis  $\{e_i^* \otimes f_j\}$
- If  $A: V \rightarrow V$ ,  $B: W \rightarrow W$ , then

$$\begin{aligned}
 A \oplus B: V \oplus W &\rightarrow V \oplus W, & A \otimes B: V \otimes W &\rightarrow V \otimes W \\
 A \oplus B(v, w) &= (Av, Bw), & A \otimes B(v \otimes w) &= Av \otimes Bw \\
 \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} & & & \text{, more involved}
 \end{aligned}$$

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### Linear Algebra 3

Given two vector spaces with bases, new vector spaces can be constructed using the direct sum ( $\oplus$ ) or tensor product ( $\otimes$ ).

## Linear Algebra 4

- Isomorphism :  $A: V \rightarrow W$  such that there exists  $B: W \rightarrow V$  with  $BA = 1_V, AB = 1_W$
- For matrix , need  $\det A \neq 0$ . Matrixes must be square  $GL_n(\mathbb{R}), GL_n(\mathbb{C})$
- Isometry or unitary map :  $\langle Av_1, Av_2 \rangle = \langle v_1, v_2 \rangle$ ,  
for any  $v_1, v_2 \in V$
- For matrix , need  $A^* A = AA^* = I$ ,  
where  $A^* = \overline{(A^T)}$ , Hermitian conjugate  
 $O(n), U(n)$

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### Linear Algebra 4

Two vector spaces are equivalent (or isomorphic) if maps can be found in either direction, such that their composition is the identity map both ways. An important related concept is that of a unitary map, which preserves inner products. This is automatically an isomorphism.

## Linear Algebra 5

- Hermitian :  $A = A^*$   $a_{ij} = \overline{a_{ji}}$
- Then  $B_t = e^{iAt}$  is unitary , for  $t \in \mathbb{R}$ , and  $\lim_{t \rightarrow 0} B_t = I$
- Diagonalization : A Hermitian map  $A : V \rightarrow V$  has an ON basis  $\{e_k\} \ni A(e_k) = \alpha_k e_k$

$$A \leftrightarrow \begin{pmatrix} \alpha_1 & 0 \\ & \ddots \\ 0 & \alpha_n \end{pmatrix}$$

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## Linear Algebra 5

A Hermitian matrix can be diagonalized. Its eigenvectors can be chosen to be orthonormal, and its eigenvalues are real.

## Linear Algebra 6

- **Frames:**  $\{e_k\}$

There are numbers  $0 < A \leq B < \infty$

such that for any  $v$  in  $V$ ,

$$A \|v\|^2 \leq \sum |\langle v, e_k \rangle|^2 \leq B \|v\|^2$$

- **Tight** if  $A = B$ ; **Exact** if we cannot remove any vector without losing completeness

$(1,0), (1,1)$  an exact frame

$(1,0), (1,0), (0,1), (0,1)$  is a tight frame,  $A=B=2$

- **Tight exact frame with  $A=B=1$  is an ON Basis**

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## Linear Algebra 6

While we will mainly deal with orthonormal bases, there is a generalization of the notion of bases that is useful: frames. A frame is a set of vectors, often too many to be a basis, such that the size of any vector in the space can be bounded above and below in terms of the inner products of that vector with the vectors in the frame. A *tight* frame is a basis, while an *exact* frame is one where the bound constants are equal. An orthonormal basis is a tight exact frame with bound constants equal to 1.

## Linear Analysis 1

- Infinite dimensions :  $l_2 = \left\{ (a_i) \mid \sum_1^\infty |a_i|^2 < \infty \right\}$  complex sequences
- $L^2(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{C} \mid \int |f|^2 < \infty\}$ , complex functions
- Also  $L^2([0,1]), L^2(\mathbb{R}^n), L^2(X), X \subset \mathbb{R}^n$
- Example ,  $X = S^1$ , circle in  $\mathbb{R}^2$
- All these spaces are isomorphic, in fact unitarily!  
That is because they all have an orthonormal basis.  
Different "realizations" of one space  $H$ , Hilbert space.

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### Linear Analysis 1

A finite dimensional space is equal to the space of  $n$  arbitrary numbers ( $n$  = dimension). In infinite dimensions, instead of dealing with spaces of arbitrary numbers, the notion of convergence enters. One has the space of infinite sequences where the sum of the norm-squared terms is finite. While there are several other natural generalizations (e.g., the functions on the real line whose norm-squared integral is finite), all of these spaces are isomorphic to each other (by a unitary map). That is because they all have orthonormal bases of the same cardinality.

## Linear Analysis 2

- $L^2([0,2]) = L^2([0,1] \cup [1,2]) = L^2([0,1]) \oplus L^2([1,2])$

$$= H \oplus H \cong H!$$

- $L^2(\mathbb{R}) = L^2\left(\bigcup_{n=-\infty}^{\infty} [n, n+1]\right) = \bigoplus_{n=-\infty}^{\infty} L^2([n, n+1]) = \bigoplus_{n=-\infty}^{\infty} H_n \cong H!$

- $L^2(\mathbb{R}^n) = L^2(\mathbb{R}) \otimes \dots \otimes L^2(\mathbb{R}) = H_1 \otimes \dots \otimes H_n \cong H!$

Although all of these spaces are really the same, their different aspects give rise to different bases.

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### Linear Analysis 2

If we denote the common space by  $H$ , the fact that the various definitions lead to the same space implies some interesting things about  $H$  under direct sum ( $\oplus$ ) and tensor product ( $\otimes$ ).

## Linear Analysis 3

- $l_2$  has standard basis  $\{e_k\}$ ,  $e_k = (0, \dots, 1, \dots, 0, \dots)$
- $L^2(\mathbb{R})$  has no natural ON basis!  
Traditionally, important ON bases arose in relation to natural Hermitian operators.
- "Particle on a loop"  $L^2(S^1)$   
 $\nabla^2 f = \lambda f$ ,      Eigenvectors of Laplace operator  
 $\{e^{2\pi i n x}\}$ ,  $n \in \mathbb{Z}$  (Integers)
- $L^2(\mathbb{R}) \xrightarrow{\sim} l_2$  ON basis  $\{e_n\}$  of  $L^2(\mathbb{R})$   
 $f \mapsto \{\langle f, e_n \rangle\}$

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### Linear Analysis 3

The sequence space has a standard basis, but the other "realizations" of  $H$  have no natural bases. Orthonormal bases frequently arise in relation to Hermitian operators. The special functions of mathematical physics are instances of this. In particular, the Fourier series basis derives from the Laplace operator on a circle (or an interval). Once an orthonormal basis is secured, the equivalence to the sequence space is constructed by taking dot products with basis vectors.

## Linear Analysis 4

- View  $H$  as  $L^2(\mathbb{R})$
- Linear map  $T: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  can be represented as an integral operator  
 $(Tf)(x) = \int k(x, y)f(y)dy,$   
 some "function"  $k(x, y)$ , called the kernel, on  $\mathbb{R}^2$
- The "function"  $k(x, y)$  may have  $\delta$ -function singularities
- Example:  $T = I$ . Then  $k(x, y) = \delta(x - y)$
- "Resolution of Id": If  $\{e_n(x)\}$  is an ON basis,  
 then for identity map  $I$ ,  $k(x - y) = \sum_n e_n(x) \overline{e_n(y)}$   
 (Physics:  $1 = \sum_n |n\rangle \langle n|$ )

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### Linear Analysis 4

In fact, every linear map on  $L^2(\mathbb{R})$  can be represented as an integral operator (although the "kernel" may have delta function singularities, e.g., the case of the identity operator). This is the function space version of the fact that any linear map can be represented by a matrix.

## Linear Analysis 5

- Hermitian:  $k(x, y) = \overline{k(y, x)}$
- LTI:  $k(x, y) = k(x - y)$ , some  $k: \mathbb{R} \rightarrow \mathbb{C}$
- Convolution:  $(f * g)(x) = \int f(x - y) g(y) dy$

So LTI systems are convolution operators.  
 $h$  = impulse response function

- Matrix: ON basis  $\{e_k\}$ ,  $k = 1, 2, \dots$   
 $T: H \rightarrow H$  has  $(a_{ij})$ ,  $Te_j = \sum_i a_{ij} e_i$

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### Linear Analysis 5

In the case of a linear “time-invariant” operator, the kernel is really a function of one variable. Such an operator is thus a convolution with a fixed function, called the impulse response function (the output of a delta function input).

# Fourier Transform 1

- Unitary Transform:  $T: H \rightarrow H$   
 $\langle Tv, Tw \rangle = \langle v, w \rangle$ , all  $v, w \in H$

Examples:  $(T_a f)(x) = f(x - a)$  translation  
 $(E_a f)(x) = e^{2\pi i a x} f(x)$  modulation  
 $(D_a f)(x) = \sqrt{a} f(ax)$  dilation

A nxn matrix,  $\det A = 1$  ( $SL_n(\mathbb{R})$ )

$T_A: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$

$(T_A f)(x) = f(Ax)$

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## Fourier Transform 1

Important examples of unitary transforms of the function space  $L^2(\mathbb{R})$  are translation, modulation and dilation.

## Fourier Transform 2

- **Fourier Transform:**

$$(F f)(\omega) = \hat{f}(\omega) = \int e^{-2\pi i x \omega} f(x) dx = \langle f, E_{\omega}(1) \rangle$$

- "Miracle"  $D\hat{f}(\omega) = (2\pi i \omega)\hat{f}(\omega)$

Turns differential equations in algebraic equations

- $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$  Parseval's Thm

- $(f * g)^{\hat{}} = \hat{f} \hat{g}$  Convolution Thm

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### Fourier Transform 2

From the definition of the Fourier transform, it is easy to see that it is unitary (Parseval's theorem). An important fact about the Fourier transform is that it turns differentiation operators into multiplication operators, simplifying the solution of differential equations. For signal analysis, the crucial fact is that the Fourier transform turns convolution into multiplication. These important facts justify the central role of the Fourier transform in analysis. We will see that certain practical considerations will lead us elsewhere, however.

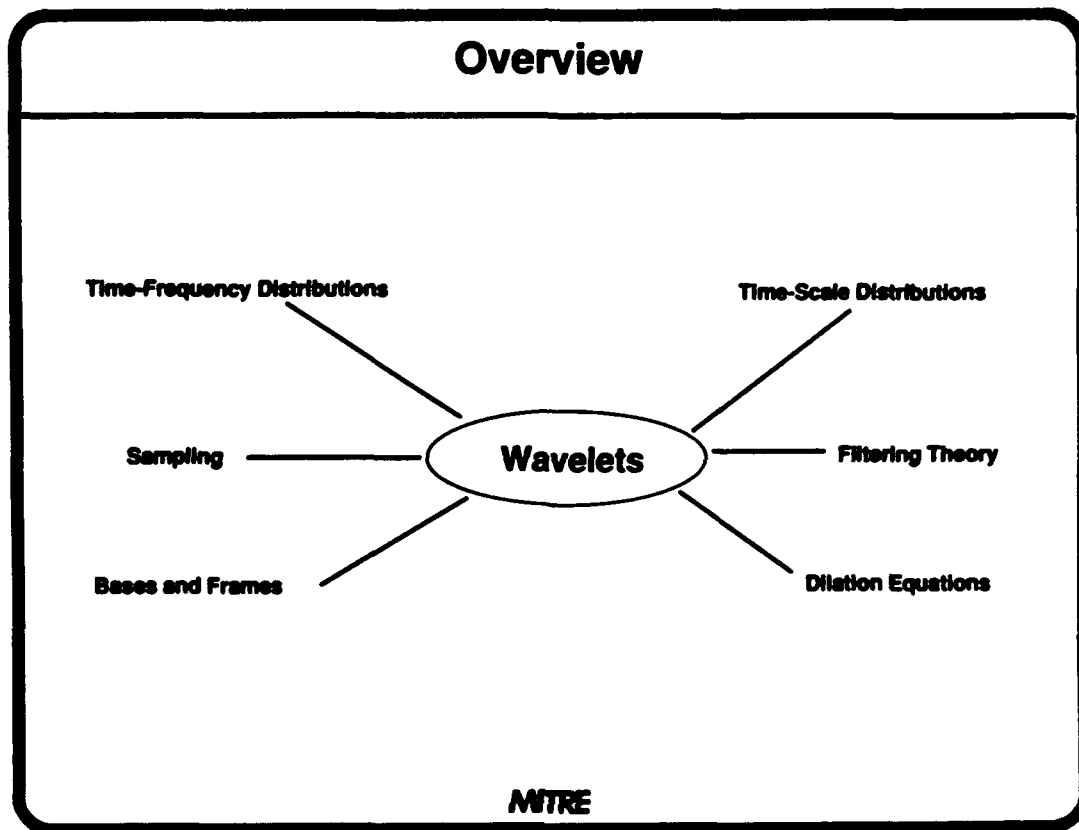
## **Theory**

- **Gabor Wavelets**
- **Affine Wavelets**
- **Time-Frequency Localization**

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### **Theory**

In this section we outline the theory of the wavelet transform as it exists today. There are basically two types of wavelets: Gabor wavelets, which relate to translations and modulations; and affine wavelets, which deal with translation and dilations. There is an essential difference between these two types of wavelets in the way they divide the time-frequency plane. This leads us into some general considerations of localization in the time-frequency plane.



### Overview

Wavelets relate to a great number of topics in signal processing. In this course, we will cover the time-frequency aspects of wavelets, along with considerations of sampling, filtering, and bases and frames.

## History

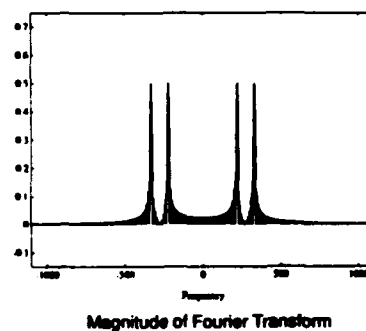
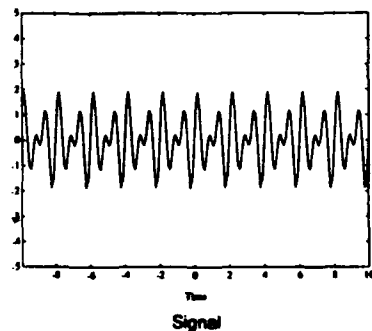
- 1910 A. Haar**
- 1920 J.L. Walsh**
- 1946 D. Gabor**
- 1977 D. Esteban, C. Galand**
- 1980-5 P. Burt, E. Adelson**  
**M.J.T. Smith, T.P. Barnwell**  
**M. Vetterli**
- 1985 A. Grossmann, J. Morlet**
- 1987 I. Daubechies, Y. Meyer, S. Mallat**

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## History

In 1910, Haar discovered an orthonormal basis constructed from a single function, subject to discrete translations and dilations. Another such orthonormal basis was not discovered until 1985, by Daubechies and others. Meanwhile, in 1946, D. Gabor laid the foundations of the time-frequency approach to bases, leading to the short-time Fourier transform. In 1977, Estaban and Galand, researchers at IBM-France, devised a filtering method into low and high passbands that were free of aliasing on reconstruction. Although an FIR filter could not be found to satisfy their criteria (a problem solved by Smith and Barnwell in 1980), they laid the foundations of the present theory of wavelets. Grossmann and Morlet constructed affine wavelet bases for good signal decomposition in the context of oil exploration. Finally, the mathematicians Daubechies, Meyer and Mallat constructed an elegant theory for this time-scale approach to bases.

## The Fourier Transform



- **Fourier theory is well-established**
  - **Decomposition of signal into frequency components (Fourier coefficients)**
  - **Basic signal components given by functions  $e^{2\pi i f t}$**

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### The Fourier Transform

To introduce the wavelet basis, recall that the Fourier transform decomposes a signal into its harmonic components. This is useful when the signal has well-defined periodicities, such as the sum of two sine waves as depicted.

## **The Fourier Transform (Concluded)**

- **Stationarity assumption – signal components fixed for all time**
  - **Fourier coefficients require knowledge of signal for all time**
  - **Poor time-frequency localization**
- **Possible alternatives**
  - **Windowed Fourier Transform (Gabor Wavelets)**
  - **General Time-Frequency Distributions**
  - **Time-Scale Distributions (Affine Wavelets)**

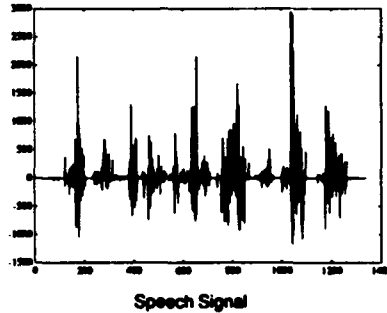
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### **The Fourier Transform (Concluded)**

In the context of stochastic processes, one requires the assumption of stationarity. While the signal may be random, statistically it has the same harmonics at all times. For signals that do not satisfy this assumption (no actual signal is stationary), so that the harmonic content is time-varying, the Fourier method is inadequate in that it does not register the time structure of harmonics. As an example, the Fourier transform of a musical piece would tell you only what notes occurred, not when they occurred.

To get around this problem, several methods have been developed. Gabor thought to apply a running “window” to the signal before doing the Fourier transform, now called the short-time Fourier transform. More generally, there are ways to represent a running spectral content of a signal, not dependent on a windowing function. These fall under the rubric of time-frequency distributions, which we will discuss later. Finally, there is the time-scale approach of the affine wavelets, which is the focus of this course.

## Non-Stationary Signal Processing



Time-dependent spectrum

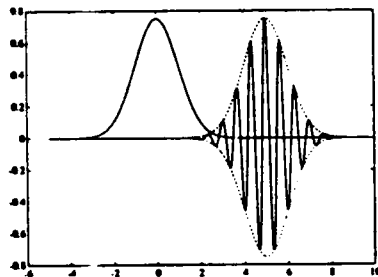
- Wavelets are a “new” tool for time-frequency analysis
- Two classes of wavelets
  - Gabor wavelets: windowed Fourier transform
  - Affine wavelets: provide multiscale decomposition

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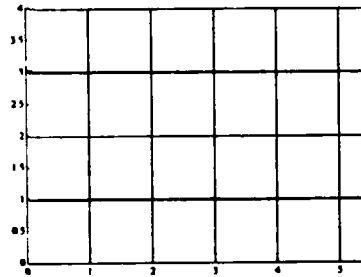
### Non-Stationary

These new wavelet tools allow the analysis of nonstationary signals, which abound in nature. A classic example is speech (or music), whose “information” is clearly in the changing spectral content.

## Gabor Wavelets



Gabor Wavelet



Time-Frequency Plane

- Window functions have fixed size and different shapes
- Window function fixes time and frequency resolution
  - Uncertainty principle dictates  $(\Delta t)(\Delta f) \geq \frac{1}{4\pi}$
- Useful for analysis of narrowband processes

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### Gabor Wavelets

To improve time-localization over the Fourier transform, one can multiply the signal by a “window” function first. Since the Gaussian function has optimal time-frequency spread according to the Uncertainty Principle (equality is achieved), Gabor chose the Gaussian as his window function. Thus, the Gaussian function, translated and modulated integral amounts, is the complete system of functions called the Gabor wavelets. Note that unlike the Fourier basis, this system is not orthogonal.

## Gabor Wavelets (Continued)

- **Basic window function**  $g$ , e.g.

$$g(t) = \chi_{[-1/2, 1/2)}(t) \quad \text{Haar}$$

$$g(t) = \pi^{-1/4} e^{-t^2/2} \quad \text{Gabor}$$

- **Energy distribution in time:**  $(\Delta t)^2 = \|tg(t)\|^2 / \|g\|^2$   
**Energy distribution in frequency:**  $(\Delta f)^2 = \|f\hat{g}(f)\|^2 / \|\hat{g}\|^2$

- **Window functions are defined by**

$$g_{a,b}(t) = e^{2\pi i a t} g(t - b)$$

$$\hat{g}_{a,b}(f) = e^{-2\pi i (f-a)b} \hat{g}(f - a)$$

- **If energy in time-frequency plane for  $g$  is centered at  $(a_0, b_0)$  then energy in time-frequency plane for  $g_{a,b}$  is centered at  $(a_0 + a, b_0 + b)$**

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## Gabor Wavelets (Continued)

The uncertainty, or spread, of a function in time and frequency is defined as above. Regardless of which function is chosen, the product of the two spreads is bounded below (Heisenberg). Furthermore, translation and modulation do not change the characteristics of spread, but only shift the center of concentration in the time-frequency plane (defined by means of the quantities  $|f|^2, |\hat{f}|^2$ ).

## Gabor Wavelets (Continued)

- Windowed transform  $W_g$  maps  $L^2(R) \rightarrow L^2(R^2)$

$$W_g s(a, b) = \int s(t) \overline{g_{a,b}(t)} dt$$

- Reconstruction formula

$$s(t) = \iint W_g s(a, b) g_{a,b}(t) da db$$

- Redundancy in continuous-parameter representation

- Some redundancy can be good
- Sampled transform can provide reconstruction of  $s$

$$s(t) = \sum_{m,n} W_g s(a_0 m, b_0 n) g_{a_0 m, b_0 n}(t)$$

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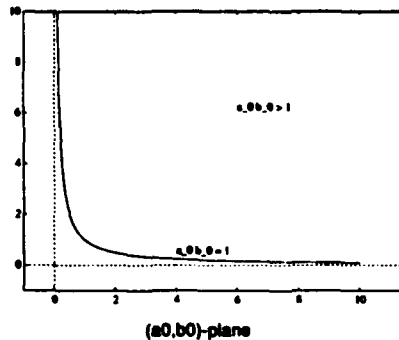
## Gabor Wavelets (Continued)

For a choice of a window function  $g$  with  $\|g\| = 1$ , the Gabor transform is given by an inner product of the signal with a translated, modulated  $g$ . The Gabor transform, as a function of the translation and modulation parameters, allows exact reconstruction of the original signal. In fact, discrete samples of the transform can also provide reconstruction, for special choice of lattice parameters  $a_0, b_0$ .

## Gabor Wavelets (Continued)

- Reconstruction depends on choice of parameters  $a_0, b_0$

$$g_{m,n}(t) = e^{2\pi i a_0 m t} g(t - b_0 n)$$



$a_0 b_0 < 1$     Frames  
 $a_0 b_0 = 1$     Bases Possible  
 $a_0 b_0 > 1$     Undersampling

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## Gabor Wavelets (Continued)

For  $a_0 b_0 \leq 1$ , reconstruction is possible ( $g$  has to be well-chosen also). The case  $a_0 b_0 = 1$  is special in that actual bases are possible, not just frames (which often have *more* vectors than necessary). For  $a_0 b_0 > 1$ , reconstruction is impossible for any  $g$ .

## Gabor Wavelets (Concluded)

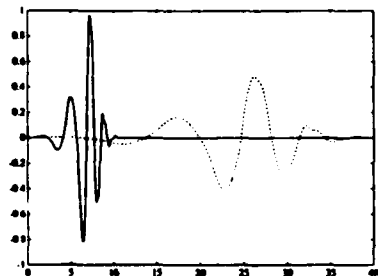
- **Uncertainty Principle dictates that**  $(\Delta t)(\Delta f) \geq \frac{1}{4\pi}$   
 $\{g_{m,n}\}$  basis (exact frame)  $\Rightarrow \|tg(t)\| \|f\hat{g}(f)\| = \infty$
- **Thus good time-frequency localization is not possible for Gabor wavelet bases (exact frames)**
- **Equivalently, good time-frequency localization requires redundancy in the Gabor transform**

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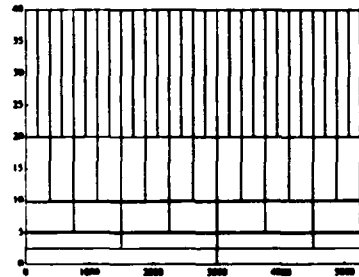
## Gabor Wavelets (Concluded)

Unfortunately, for such a Gabor frame to be a basis, the window function must have poor time-frequency localization properties—either the time or frequency spread diverges. In particular, the Gaussian window does not give rise to any basis, since its time and frequency spreads are finite (in fact optimal). In general, good time-frequency localization with Gabor wavelets can be achieved only by using frames.

## Affine Wavelets



Affine Wavelet



Time-Frequency Plane

- **Window functions are obtained by dilating and translating a single function**
  - Functions have different sizes and same shape
- **Provides for multiscale analysis**
  - Good time resolution at high frequencies
  - Good frequency resolution at low frequencies

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## Affine Wavelets

Affine wavelets are constructed by translations and *dilations* of a single functions. These give rise to a different tiling of the time-frequency plane when considering discrete bases. In essence, affine wavelets achieve good time resolution at high frequencies, and good frequency resolution at low frequencies (at the expense of good frequency or time resolution, respectively). For numerous applications, this choice in the time-frequency trade-off is advantageous.

## Affine Wavelets (Continued)

- **General Theory**

$\psi \in L^2(R)$  fixed;      set  $\psi_{a,b}(t) = a^{1/2}\psi(at - b)$

$W_\psi : L^2(R) \rightarrow L^2(R^2)$

$W_\psi s(a, b) = \int s(t) \overline{\psi_{a,b}(t)} dt$

- **For the reconstruction of  $s$**

$$s(t) = C_\psi^{-1} \iint W_\psi s(a, b) \psi_{a,b}(t) \frac{da db}{a}$$

**we need an admissibility condition**

$$C_\psi = \frac{1}{2\pi} \int |\hat{\psi}(f)|^2 / |f| df < \infty$$

**so that  $\hat{\psi}(0) = 0$**

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## Affine Wavelets (Continued)

The affine wavelet transform is given by an inner product of the signal with a translated, dilated wavelet. It also allows reconstruction of the original signal, under mild conditions on the wavelet function (it has zero mean).

## Affine Wavelets (Continued)

- **Discrete Version:**  $\psi_{m,n}(t) = a_0^{m/2} \psi(a_0^m t - b_0 n)$
- **Now**

$$\hat{\psi}_{m,n}(t) = a_0^{-m/2} e^{-2\pi i f b_0 n / a_0^m} \hat{\psi}(f / a_0^m)$$

**so that the parameter  $m$  determines the resolution in time and frequency while the parameter  $n$  locates the center of energy within each scale**

**$m \gg 0$  : good time resolution**

**$m \ll 0$  : good frequency resolution**

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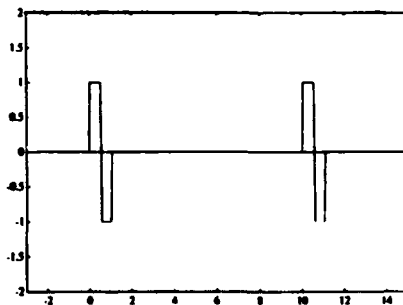
## Affine Wavelets (Continued)

Again, discrete samples of the wavelet transform can suffice for reconstruction. The dilation parameter effectively controls the time-frequency characteristics of the wavelet.

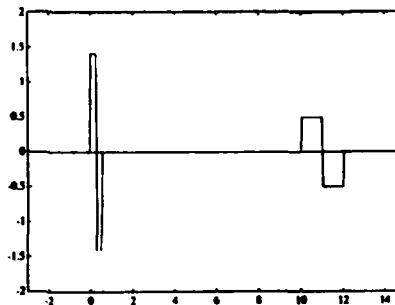
## Affine Wavelets (Concluded)

- Consider the Haar example:  $a_0 = 2, b_0 = 1$

$$\psi = \psi_{0,0} = \chi_{[0,1/2)} - \chi_{[1/2,1)}$$



$\psi_{0,0}$  and  $\psi_{0,10}$



$\psi_{1,0}$  and  $\psi_{-1,5}$

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## Affine Wavelets (Concluded)

As a basic example, the Haar wavelet and certain of its translated and dilated cousins are depicted.

## Wavelet Transform

$$W_{\psi}s(a, b) = \int_{-\infty}^{\infty} s(t) \overline{\psi_{a,b}(t)} dt$$

### Gabor

$$a = m, \quad b = n$$

$$\psi_{m,n}(t) = e^{2\pi i m t} g(t - n)$$

### Affine

$$a = 2^m, \quad b = n$$

$$\psi_{m,n}(t) = 2^{m/2} \psi(2^m t - n)$$

$$s(t) = \sum_{n,m=-\infty}^{\infty} W_{\psi}s(m, n) \psi_{m,n}(t)$$

↑

wavelet transform

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## Wavelet Transform

In summary, both the Gabor and Affine wavelet transforms are given by inner product of the signal against a two-parameter family of functions. Both allow discrete sampling to provide reconstruction.

## Multiresolution Analysis

**A multiresolution analysis for finite energy signals  $L^2(\mathbb{R})$  is a collection of closed subspaces  $\{V_m\}$  satisfying the properties**

- $V_m \subset V_{m+1}$
- $\cap V_m = \{0\}$
- $\overline{\cup V_m} = L^2(\mathbb{R})$
- $s(t) \in V_m \Leftrightarrow s(2t) \in V_{m+1}$
- **There exists  $\varphi$  so that  $\{\varphi(t - n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis for  $V_0$**

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### Multiresolution Analysis

Essentially, every orthonormal wavelet basis can be derived from a construction due to Mallat and Meyer called multiresolution analysis. Assume that the function space  $L^2(\mathbb{R})$  has a sequence of nested subspaces, whose elements are related by dilation by powers of 2. Furthermore, assume that one such space, say  $V_0$ , has a function  $\varphi$  whose integer translates are in  $V_0$ , and form an orthonormal basis.

## Multiresolution Analysis (Continued)

- At each resolution level  $V_{m+1} = V_m \oplus W_m$

- $V_m = \{s(2^m t) : s \in V_0\}$   
 $W_m = \{s(2^m t) : s \in W_0\}$

- We thus obtain a decomposition of  $L^2(\mathbb{R})$ :

$$\begin{aligned} L^2(\mathbb{R}) &= V_m \oplus W_{m+1} \oplus W_{m+2} \oplus \cdots \\ &= \oplus W_m \end{aligned}$$

Here,  $V_m$  is the resolution at level  $m$  and the remainder  $W_{m+1} \oplus \cdots$  contains the high frequency components needed to generate all of  $L^2(\mathbb{R})$

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## Multiresolution Analysis (Continued)

Since  $V_m \subset V_{m+1}$ , one can introduce the orthogonal complement space  $W_m$ , so that  $V_{m+1} = V_m \oplus W_m$ . Since  $\bigcap_m V_m = 0$ , and  $\overline{\bigcup_m V_m} = L^2(\mathbb{R})$  one has that  $\bigoplus_m W_m = L^2(\mathbb{R})$ . That is, any square integrable function can be written as a sum of components that live in the  $W_m$ . In fact, the spaces  $W_m$  are spanned by the translates of a fixed dilate of a single function,  $\psi$ .

## Multiresolution Analysis (Concluded)

- Since  $V_0 \subset V_1$  there are coefficients  $\{c_n\}$  so that

$$\varphi(t) = \sum_n c_n \varphi(2t - n)$$

- A basis for  $W_0$  is given by  $\{\psi(t - n)\}_{n \in \mathbb{Z}}$  where  $\psi$  is given explicitly in terms of  $\varphi$

$$\psi(t) = \sum_n (-1)^n c_{1-n} \varphi(2t - n)$$

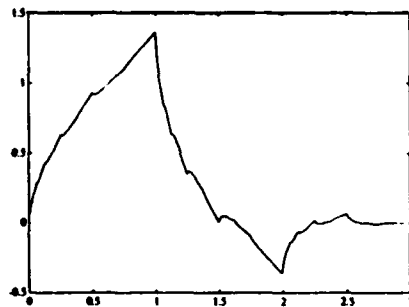
- $\varphi, \psi$  satisfy  $\int \varphi = 1$  and  $\int \psi = 0$

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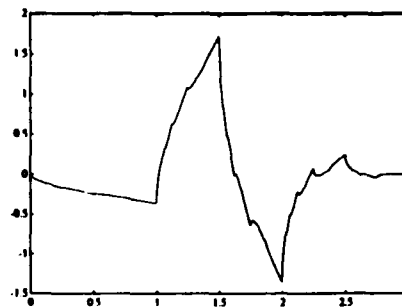
## Multiresolution Analysis (Concluded)

Given this setup, the function  $\varphi$  satisfies a recursion relation called a dilation equation. The coefficients, or taps, in the relation play an important role. A formula exists for constructing a function  $\psi$  from  $\varphi$  and its cousins, such that translates and dilates of  $\psi$  form an orthonormal basis.  $\psi$  automatically satisfies the requirement of zero mean.

### Example: Daubechies 4-Tap Wavelet



Scaling Function



Wavelet

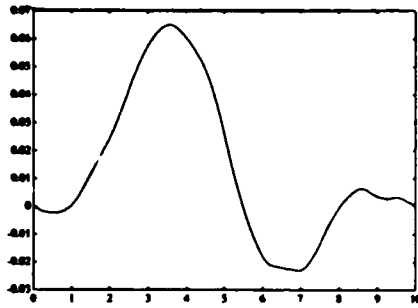
$$c_0 = \frac{1+\sqrt{3}}{4} \quad c_1 = \frac{3+\sqrt{3}}{4} \quad c_2 = \frac{3-\sqrt{3}}{4} \quad c_3 = \frac{1-\sqrt{3}}{4}$$

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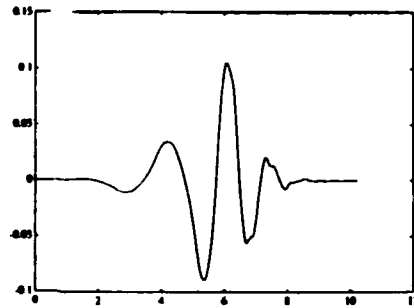
### Example: Daubechies 4-Tap Wavelet

The Haar function satisfies a 2-tap dilation equation. Daubechies (75 years later) discovered a function that satisfies a 4-tap relation. The functions  $\phi$ ,  $\psi$  (called scaling function and wavelet, respectively) in this case are very unusual in that they are continuous but not differentiable.

### Example: Daubechies 10-Tap Wavelet



Scaling Function



Wavelet

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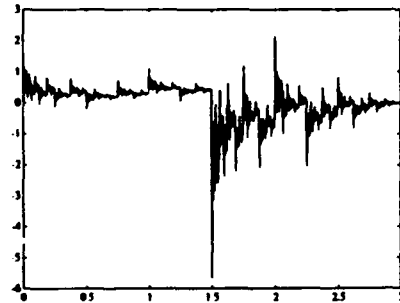
### Example: Daubechies 10-Tap Wavelet

Allowing more taps in the dilation equation permits smoother solutions, though still not infinitely differentiable.

### Example: Non-Continuous 4-Tap Wavelet



Scaling Function



Wavelet

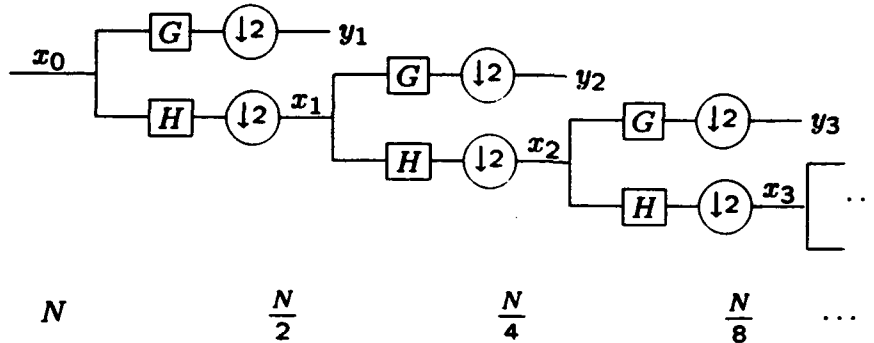
$$c_0 = \frac{1+\sqrt{2}}{2} \quad c_1 = \frac{1}{2} \quad c_2 = \frac{1-\sqrt{2}}{2} \quad c_3 = \frac{1}{2}$$

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### Example: Non-Continuous 4-Tap Wavelet

At the other extreme, even non-continuous wavelets exist (the Haar function is a simple 2-tap example).

## Fast Wavelet Transform



- $G$  highpass filter  $\{(-1)^n c_{1-n}/\sqrt{2}\}$   
 $H$  lowpass filter  $\{c_n/\sqrt{2}\}$
- Perfect reconstruction of  $x_0$  from  $x_p, y_1, y_2, \dots, y_p$  with order  $N$  algorithm

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### Fast Wavelet Transform

The coefficients in the relations for  $\phi, \psi$  give rise to FIR filters  $H, G$ , which serve as lowpass and highpass filters, respectively. A sampled signal of length  $N$  is high and lowpass filtered and subsampled. The resulting lowpass signal, of length  $N/2$ , is again high and lowpass filtered, and so on, until the process stops. The resulting set of highpass signals, plus one lowpass term, allow exact reconstruction of the original signal. The whole process is  $O(N)$ .

## Fast Wavelet Transform (Concluded)

- **Analysis**

Data	$x_0[n]$
LP Filter Coefficients	$h[n] = c[n]/\sqrt{2}$
HP Filter Coefficients	$g[n] = (-1)^n c[1-n]/\sqrt{2}$
Smoothed Data	$x_1[n] = \sum_k h[2n-k]x_0[k]$
Detail Data	$y_1[n] = \sum_k g[2n-k]x_0[k]$

- **Synthesis**

$$x_0[n] = \sum_k x_1[k]\bar{h}[k] + y_1[k]\bar{g}[k]$$

where

$$\bar{h}[k] = h[2k-n] \quad \text{and} \quad \bar{g}[k] = g[2k-n]$$

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## Fast Wavelet Transform (Concluded)

This slide gives a synopsis of the digital wavelet analysis and reconstruction formulas. Note that the highpass filter is directly constructed from the lowpass one.

### Daubechies' List

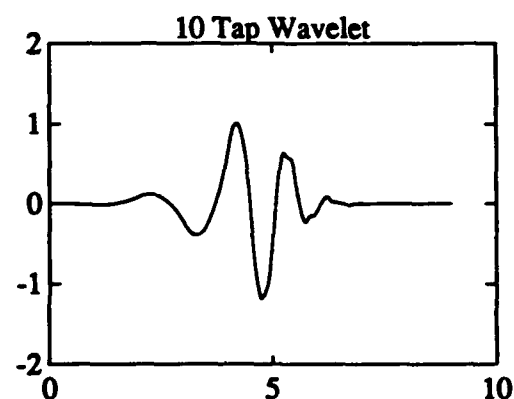
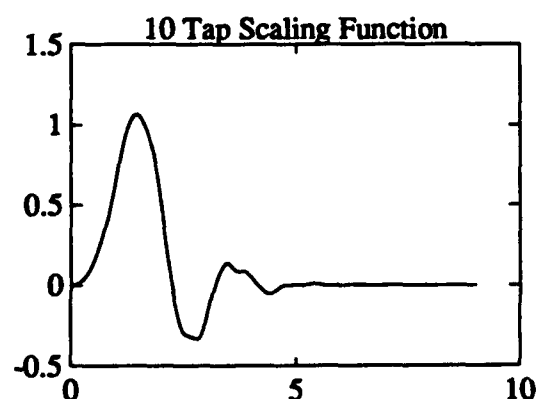
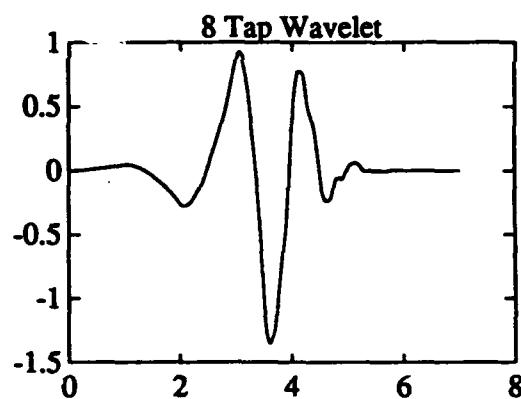
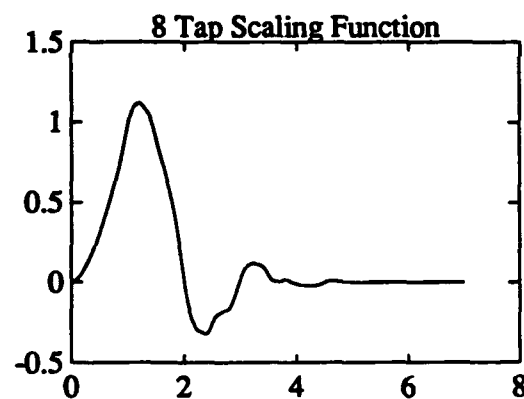
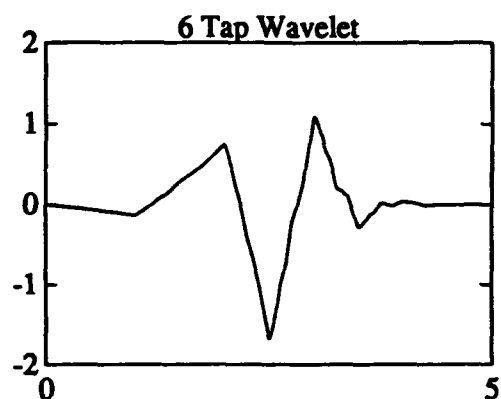
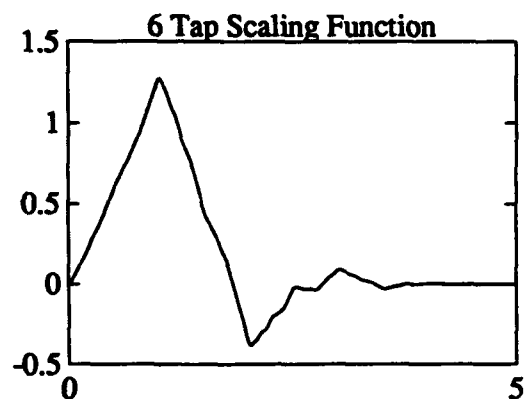
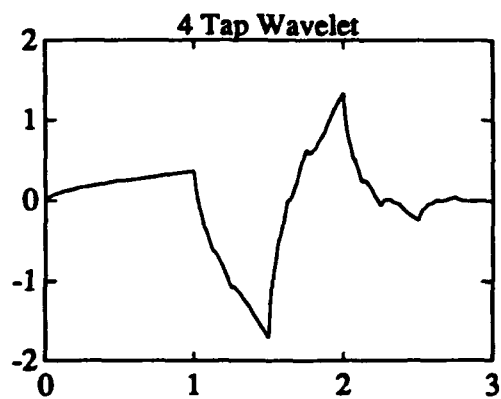
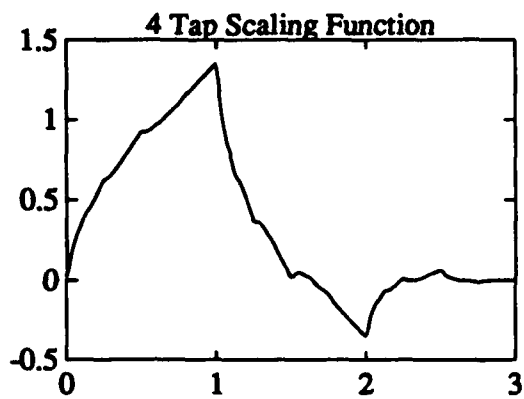
In 1988, Daubechies published a list of useful wavelet lowpass filters, which she labeled as  $Nh_n$  for integers  $N$  (unrelated to signal length;  $2N$  is the number of taps). Here we present the 4, 6, 8, 10, 12, and 14 tap filters.

	$n$	$Nh_n$
N = 2	0	.4829629131445341
	1	.8365163037378077
	2	.2241438680420134
	3	-.1294095225512603
N = 3	0	.3326705529500825
	1	.8068915093110924
	2	.4598775021184914
	3	-.1350110200102546
	4	-.0854412738820267
	5	.0352262918857095
N = 4	0	.2303778133088964
	1	.7148465705529154
	2	.6308807679398587
	3	-.0279837694168599
	4	-.1870348117190931
	5	.0308413818355607
	6	.0328830116668852
	7	-.0105974017850690
N = 5	0	.1601023979741929
	1	.6038292697971895
	2	.7243085284377726
	3	.1384281459013203
	4	-.2422948870663823
	5	-.0322448695846381
	6	.0775714938400459
	7	-.0062414902127983
	8	-.0125807519990820
	9	.0033357252854738

	$n$	$Nh_n$
<b>N = 6</b>	0	.1115407433501095
	1	.4946238903984533
	2	.7511339080210959
	3	.3152503517091982
	4	-.2262646939654400
	5	-.1297668675672625
	6	.0975016055873225
	7	.0275228655303053
	8	-.0315820393174862
	9	.0005538422011614
	10	.0047772575109455
	11	-.0010773010853085
<b>N = 7</b>	0	.0778520540850037
	1	.3965393194818912
	2	.7291320908461957
	3	.4697822874051889
	4	-.1439060039285212
	5	-.2240361849938412
	6	.0713092192668272
	7	.0806126091510774
	8	-.0380299369350104
	9	-.0165745416306655
	10	.0125509985560986
	11	.0004295779729214
	12	-.0018016407040473
	13	.0003537137999745

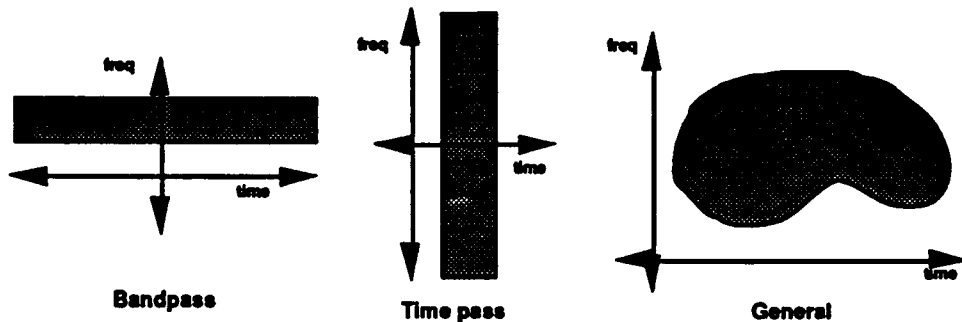
### **Daubechies' List 2**

**The graphs of some of the scaling and wavelet functions from this list are depicted here. Note that the functions are increasingly smoother down the list.**



## Time Frequency Localization via the Weyl Correspondence

- Technique for localizing a signal in a region of the time-frequency (phase) plane
- Generalizes bandpass and time pass filters to time-varying bandpass filters
- Depends on a time-frequency joint distribution and a region in the phase plane



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### Time Frequency Localization via the Weyl Correspondence

We now consider in more detail the notion of time-frequency localization of signals. The Fourier and wavelet bases gave rise to time-frequency tilings with a certain structure. Projecting a signal onto the span of a subset of these basis vectors would amount to projecting onto a subset of the time-frequency plane. While the subsets involved had a rectangular structure previously, we will now study such projection operators that relate to arbitrary domains.

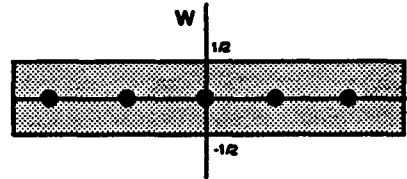
To begin with, projecting onto a subset of the time-frequency plane is a common operation in filtering theory. For example, a bandpass filter projects onto a strip domain parallel to the time axis. Similarly, timepass filtering (restricting the signal to a time domain) can be viewed as projecting onto a strip parallel to the frequency axis. But how can one project onto an arbitrary domain? And which functions are well-localized in such domains?

## Preliminaries

### Sampling Theorem

$$f(x) = \sum f(n) \operatorname{sinc}(\pi(x-n))$$

for  $f(x)$  bandlimited to  $[-1/2, 1/2]$



### Heisenberg Uncertainty

$$\|Xf\| \|Df\| \geq 1/4 \pi \|f\|^2$$

$$D = 1/2 \pi i \, d/dx$$

Extremals:

$$\phi(x) = 2^{1/4} \exp(-\pi x^2)$$

$$\exp(2\pi i a x) \phi(r(x-b))$$

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## Preliminaries

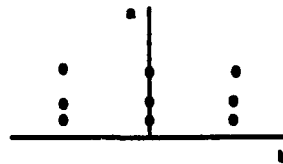
As a preliminary, consider bandlimited function. The sinc function is bandlimited. In fact, its integer translates form an orthonormal basis for bandlimited functions. Even more, the coefficients involved in expanding any bandlimited function in this basis are actually the values of the function at the integer points (Sampling Theorem). Note that one has one basis function per unit area in the band domain—this is the Nyquist density of states.

The bandlimited sinc functions are analytic functions, making them very special. However, though limited in frequency, they have very slow decay in time ( $1/x$ ). Functions that are well-localized in both time and frequency (e.g., the product of their time and frequency spreads is small) tend to be even better. For example, the extremals of the Uncertainty Principle are the (modulated, dilated, and translated) Gaussians, which have exponential decay in time and frequency, in addition to being analytic.

## Wavelet Transforms

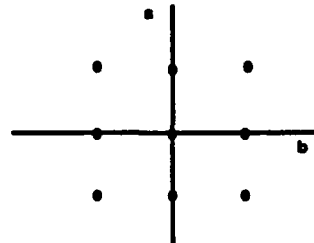
- **Affine wavelet transform**

$$T_f(a,b) = \int f(x) \overline{\psi_a} g(a x - b) dx$$



- **Gabor (W-H wavelet) transform**

$$G_f(a,b) = \int f(x) \exp(2\pi i a x) g(x - b) dx$$

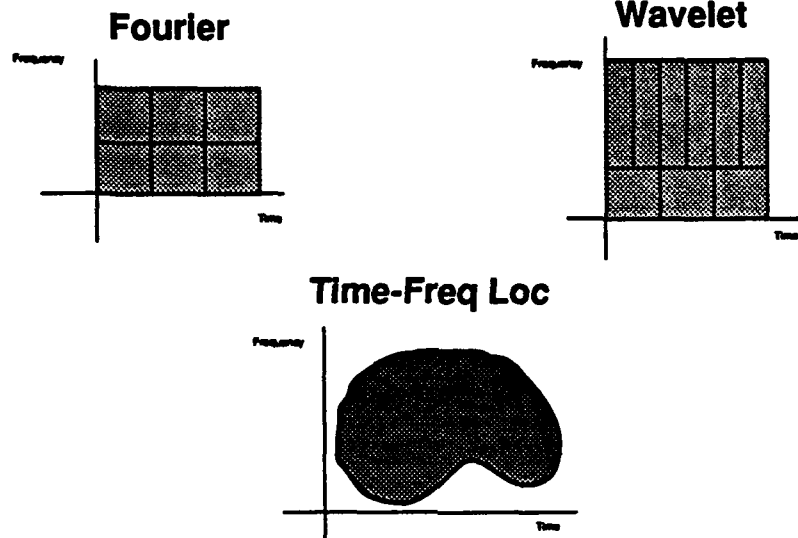


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## Wavelet Transforms

For square-integrable signals that are not necessarily bandlimited, there is still a discrete representation available, now in the Affine or Gabor Wavelet Transform. In this case, the sampling is on a two-dimensional lattice (the affine case is also on a lattice, when  $\log a$  is used as a parameter instead of  $a$ ).

## Time-Frequency Decomposition

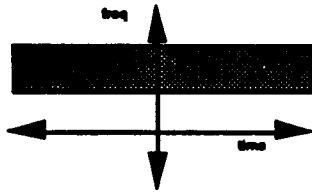


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### Time-Frequency Decomposition

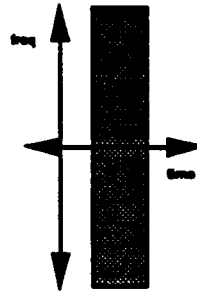
In terms of time-frequency concentration, the Fourier and wavelet bases have rectangular domains of support. In the Fourier case, the rectangles are of identical shape, while in the wavelet case they vary, though still of the same area. To find basis functions that have more general support domains, we study a class of time-frequency localization operators.

## Linear Filters



**Bandpass**

$$B: f \rightarrow (\hat{f}\chi_j)^\vee$$



**Time pass**

$$A: f \rightarrow f\chi_j$$

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## Linear Filters

The simplest time-frequency localization operators are the projections onto time or frequency pass regions, as defined above.

## Prolate Spheroidal Wavefunctions

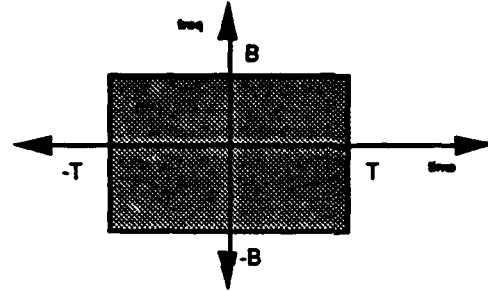
**Landau, Pollack and Slepian (1961)**

For  $f$  with  $\|f\| = 1$ ,

$$\max \|ABf\|$$

Equivalently, maximize:

$$\begin{aligned}\langle ABf, ABf \rangle &= \langle ABf, Bf \rangle \\ &= \langle BABf, f \rangle\end{aligned}$$



- Thus, seek maximum eigenvalue (and eigenfunction) of the positive, self-adjoint operator  $BAB$ .
- Succeeding eigenvalues are maximal in the space orthogonal to the previous eigenfunctions. Get the well known prolate spheroidal wavefunctions.

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### Prolate Spheroidal Wavefunctions

In 1961, Landau, Pollack, and Slepian, at Bell Labs., defined a localization operator for a rectangular region in time-frequency space. Their operator was just a concatenation of the bandpass, timepass, and bandpass operators. It emerged that their operator commuted with a Laplace type operator, whose solutions were well-known: the prolate spheroidal wavefunctions.

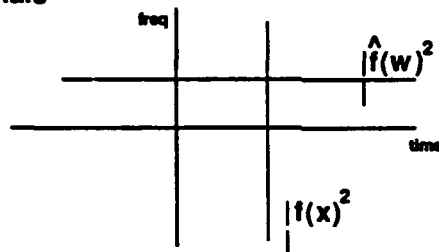
## Wigner Distribution

**Time-freq. shift:**  $\rho(a,b)f(x) = \exp(\pi i a b) \exp(2\pi i a x) f(x+b)$

**Ambiguity Fn.:**  $A(f,g)(a,b) = \langle \rho(a,b)f, g \rangle$

**Wigner Dist.:**  $W(f,g) = A(f,g)^*$

**energy representation: satisfaction of marginals**



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### Wigner Distribution

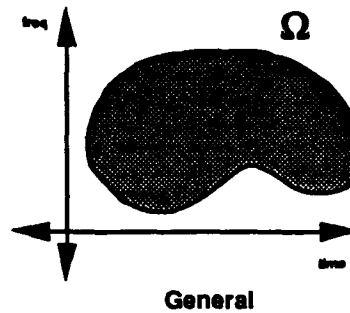
To build an operator corresponding to more general domains, we have to proceed in a different way. First, consider the Wigner distribution, which is nothing else than the Fourier transform of the Ambiguity function. The Wigner distribution of a signal is essentially a distribution of the signal's energy in the time-frequency plane. The integral of the Wigner distribution over the plane yields the energy of the signal, while integrals along lines parallel to the axes yield the power and the power spectral density, respectively.

## Weyl Correspondence

Given a symbol function  $S(w,t)$ ,  
define an operator  $L_S$  by

$$\langle L_S f, g \rangle = \int S(w,t) W(f,g)(w,t) dw dt$$

Example:  $S = \chi_\Omega$



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## Weyl Correspondence

Using the Wigner distribution, H. Weyl defined an operator as above for any "symbol" function  $S$ . We are interested in the case when  $S$  is the characteristic function of a set  $\Omega$  (i.e.,  $S(w, t) = 1$  if  $(w, t) \in \Omega$ , 0 else).

## Cohen's Class Operators

Cohen's class distribution:

$$P(f,g)(w,t) = \int W(f,g)(a,b) s(w-a, t-b) da db$$

Cohen's class operator  $L_S^P$ :

$$\langle L_S^P f, g \rangle = \int S(w,t) P(f,g)(w,t) dw dt$$

$$L_S^P = L_{S'} \quad S'(w,t) = \int S(w,t) s_-(a-w, b-t) da db$$

$$s_-(a,b) = s(-a, -b)$$

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### Cohen's Class Operators

In fact, such a procedure for defining operators can be generalized from the Wigner distribution to any one of the so-called Cohen's class of time-frequency distributions. However, such a generalization is illusory, and merely amounts to a change of the symbol function. The utility of working with distributions other than the Wigner distribution is mainly in the interpretation.

## Daubechies Operator

$$\langle D_S f, g \rangle = \int S(w, t) D(f, g)(w, t) dw dt$$

$$D(f, g)(w, t) = \langle f, \rho(-t, w) \phi \rangle \langle \rho(-t, w) \phi, g \rangle \quad \phi = 2^{1/4} \exp(-\pi x^2)$$

$$D(f, f)(w, t) = |\langle f, \rho(-t, w) \phi \rangle|^2 \quad \text{the spectrogram}$$

Realization as a Cohen's class operator

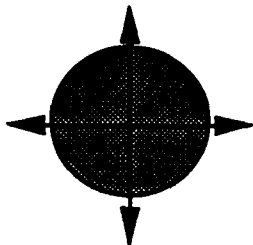
$$s(w-a, t-b) = W(\rho(-b, a) \phi, \rho(-b, a) \phi) = 2 \exp(-2\pi[(w-a)^2 + (t-b)^2])$$

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### Daubechies Operator

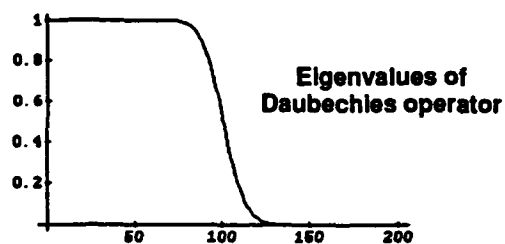
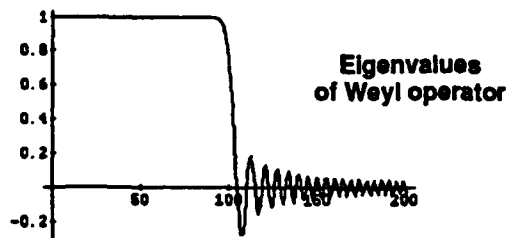
In 1988, Daubechies defined such a localization operator, in terms of the "spectrogram" distribution, which is the modulus-squared of the short-time Fourier transform with a Gaussian window. The spectrogram is a well-known tool in speech processing.

## Weyl vs. Daubechies



**Localization on a  
ball of area 100**

**Eigenfunctions =  
Hermite functions**

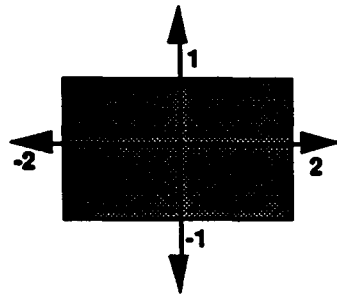


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### Weyl vs. Daubechies

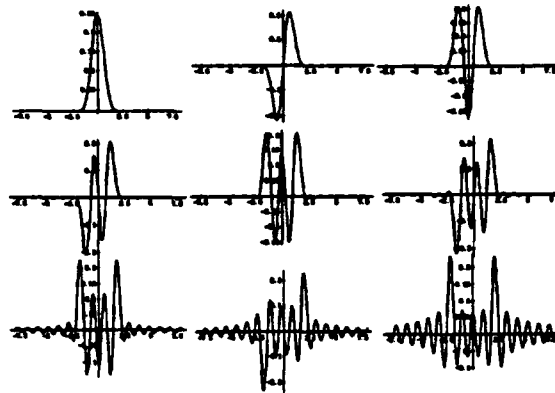
In the case of localization onto a circular disc, it turns out that the eigenfunctions for both the Weyl and Daubechies operators are identical: the Hermite functions. In this case, one can compare the eigenvalues. While there is a marked drop-off near 100 (the area) in both cases, there are two differences. The drop-off is faster in the Weyl case, but there is a Gibbs-like ringing near the zero eigenvalue. Note that for numerical purposes, the Weyl operator allows a sharper focus of energy, if projecting onto eigenvectors with eigenvalues  $> \frac{1}{2}$ .

## Prolate Spheroidal Wavefunctions

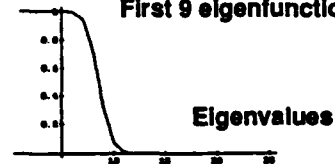


Localization on a 4x2 rectangle

Eigenfunctions =  
Prolate spheroidal  
wavefunctions



First 9 eigenfunctions

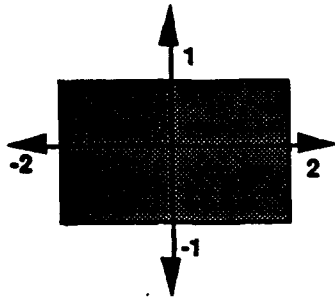


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## Prolate Spheroidal Wavefunctions

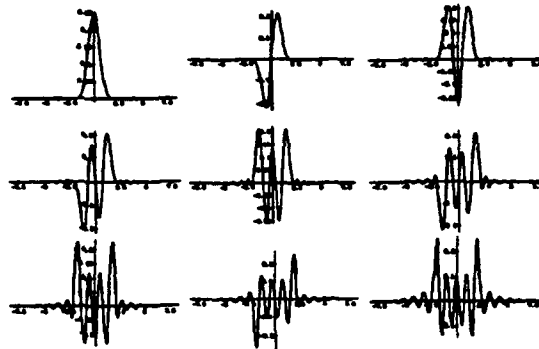
The next three slides consider projecting onto a rectangle, of area = 8. Here the first nine prolate spheroidal wavefunctions are presented. Because these functions are exactly bandlimited, they have a slow drop-off in time (e.g.,  $1/x$ ), as evidenced by the last three functions.

## Weyl Eigenfunctions

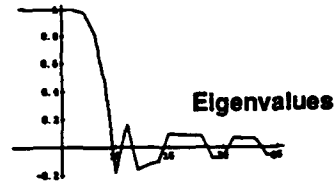


Localization on a 4x2 rectangle

Eigenfunctions =  
New Weyl solutions



First 9 Weyl eigenfunctions



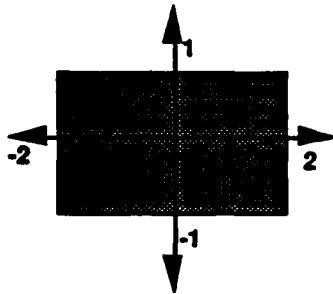
Eigenvalues

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## Weyl Eigenfunctions

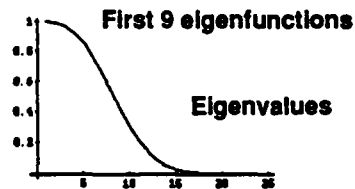
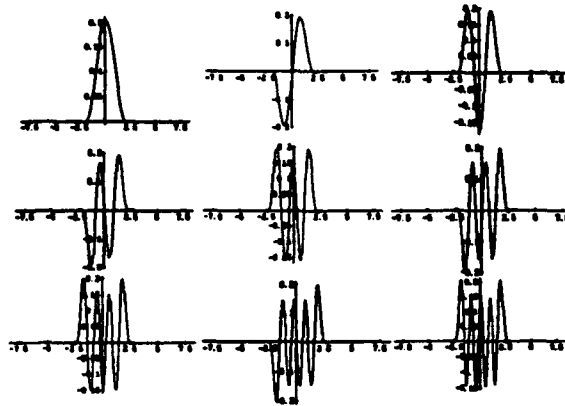
When using the Weyl correspondence, one finds eigenfunctions with a very similar appearance to the prolate functions, although the drop-off in time is better. MITRE research mathematicians were able to derive bounds on the rate of decay of the solutions.

## Daubechies Eigenfunctions



Localization on a 4x2  
rectangle

Eigenfunctions =  
New Daubechies  
solutions



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## Daubechies Eigenfunctions

In the Daubechies case, the solutions appear similar (again), but with even better decay. Note that by contrast, the eigenvalues decay slowly, indicating that the total energy is spread to many modes.

## **New Results (Wigner Distribution Case)**

- **For localization onto a general bounded domain with a piecewise (once) differentiable boundary, we prove that:**
  - **Wigner solutions are analytic**
  - **They have exponential decay, for any positive exponent**
  - **Their Fourier transforms satisfy identical conditions**
  - **Eigenvalues decay at the rate of  $n^{-3/4}$**

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### **New Results (Wigner Distribution Case)**

The qualitative observations made earlier are confirmed by theorems, proved in collaboration with Jayakumar Ramanathan of E. Michigan University, that the eigenfunctions of the Weyl operator are analytic and have exponential decay, under mild hypotheses regarding the domain of localization. [J. Ramanathan and P. Topiwala, "Time-Frequency Localization and the Weyl Correspondence," *SIAM Journal of Mathematical Analysis*, September 1993.]

## New Results (Spectrogram Case)

- For localization onto a general bounded measurable domain, we prove that:
  - Spectrogram solutions are analytic and of quadratic exp. decay
  - They and their derivatives decay faster than  $1/|x|$  to any power
  - Their Fourier transforms satisfy identical conditions
  - Eigenvalues decay faster than  $1/n$  to any power
  - The number of eigenvalues above a fixed number  $\gamma$ ,  $0 < \gamma < 1$ , is asymptotically equal to the area of the domain, upon rescaling the domain – Nyquist density.

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## New Results (Spectrogram Case)

For the spectrogram case (Daubechies operator), the results are even stronger: the solutions are analytic and have quadratic exponential decay, under very general conditions on the domain. We also have a result along the lines of Nyquist density: for a given domain, the number of eigenvalues greater than a fixed number  $\gamma$ ,  $0 < \gamma < 1$ , is asymptotically equal to the area, as the domain is dilated radially. These sets of results confirm the intuition that signals that are well-localized in the time-frequency plane satisfy strong regularity and decay conditions. Furthermore, once found, they lead to simple projection operators, by projecting onto the span of the eigenfunctions with eigenvalues greater than some cutoff (e.g.,  $\frac{1}{2}$ )

[J. Ramanathan and P. Topiwala, "Time-Frequency Localization and the Gabor Transformation," *Journal of Applied and Computational Harmonic Analysis* (to appear).]

## **Applications**

- **1-D Signal Filtering**
- **1-D Transient Detection**
- **Image Compression**

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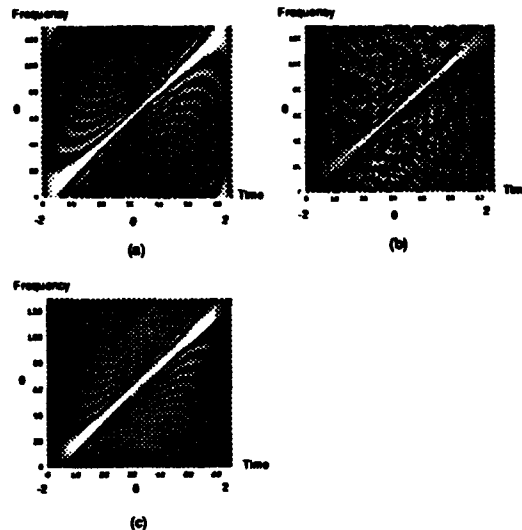
### **Applications**

**We now consider some applications of the ideas and tools developed so far. We will consider only three topics: noise and interference filtering, transient detection, and image compression.**

## Noise Filtering

**Example of noise filtering from a linear FM (chirp) signal**

**Wigner distribution of a) chirp signal; b) chirp in 0 dB noise; c) Weyl filtered noisy signal**



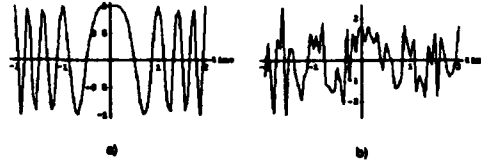
**MITRE**

### Noise Filtering

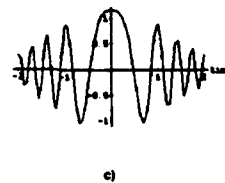
In this example, the Wigner distribution of a chirp signal is given in (a). (b) is the Wigner distribution of the chirp with 0 dB Gaussian noise added. Note that while the Wigner distribution of the chirp is a straight line, corresponding to the slope of the chirp, the noise has energy distributed all over the time-frequency plane. (c) is the Wigner distribution of the noisy signal after it has been Weyl filtered, using a narrow strip along the chirp.

## Noise Filtering (cont'd)

Plots of signals



a) chirp signal;  
b) chirp in 0 dB noise;  
c) Weyl filtered noisy  
signal

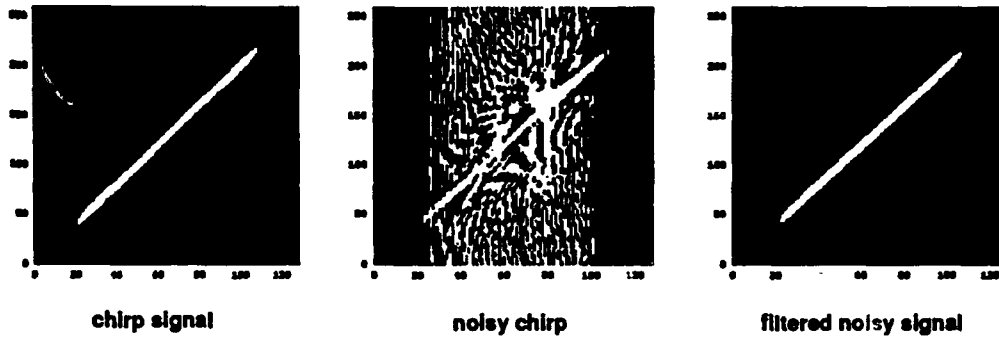


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### Noise Filtering (Continued)

This slide shows the actual signals. Note that while (c) is not an exact reconstruction of (a), it is a remarkable improvement on (b).

## Noise Filtering (final)



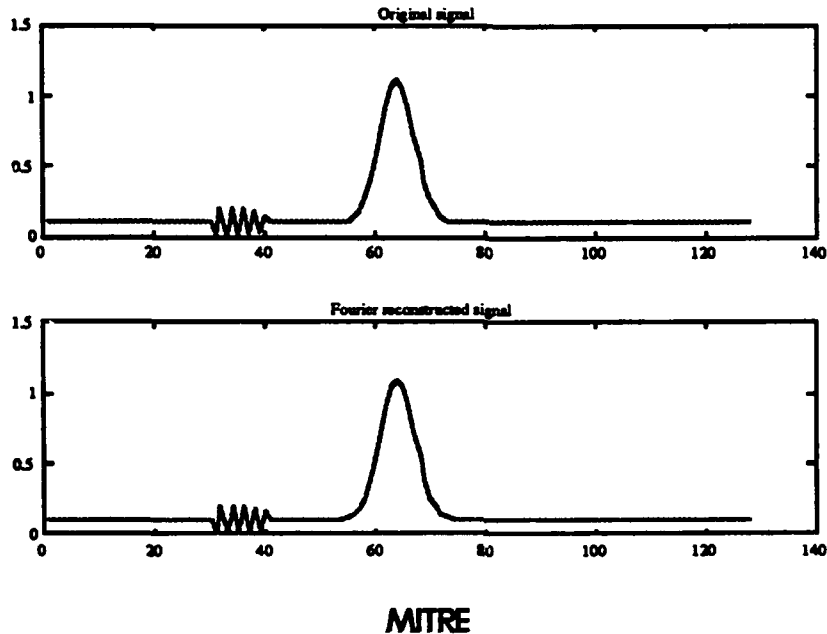
- Enhanced Wigner distribution Intensity plots using windowing of signals ( $\cos^2$ ).

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## Noise Filtering (Final)

The ringing effect in the Wigner plots can be eliminated by simple windowing of the signal—here a  $\cos^2 x$  window is used.

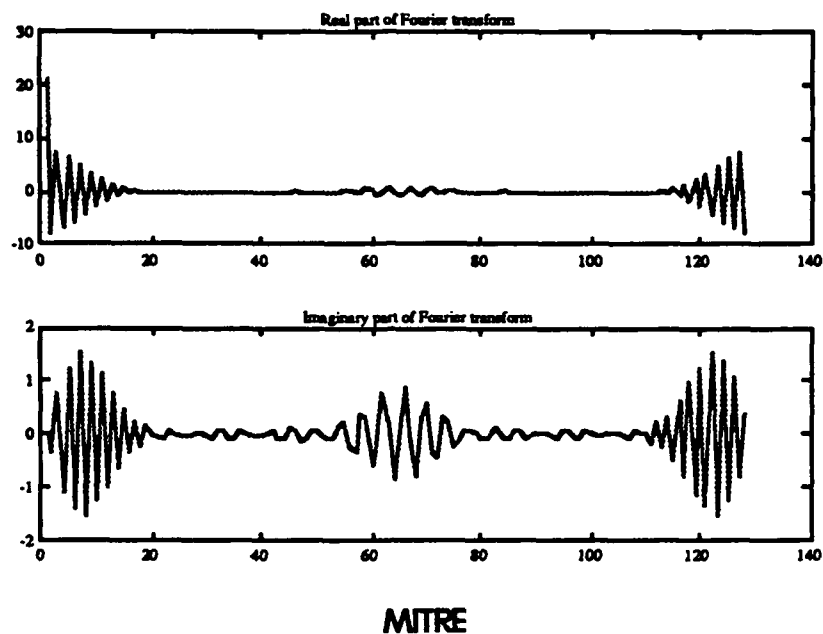
## 1-D FT Signal, $R=1$



1-D FT Signal,  $R=1$

In this example, we will consider detection of a high-frequency low magnitude transient signal in a background smooth process. Here a Gaussian pulse is used as background, and a high-frequency sinusoid as the interesting transient. We will compare the effects of using FFT vs. wavelet processing, in particular compression, on the detectability of the transient. In this slide, the original signal is reconstructed exactly under FFT processing (FFT and IFFT applied), with compression factor  $R=1$  (no compression). Note that compression is considered here as a method of feature extraction.

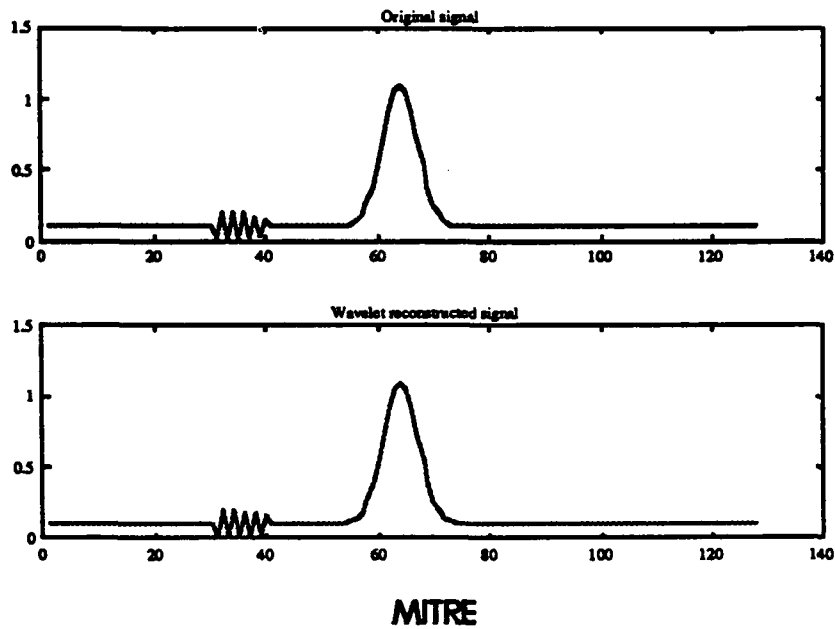
## 1-D FT Decomposition, R=1



### 1-D FT Decomposition, R=1

Here the real and imaginary parts of the Fourier transform (FFT) are displayed. The high frequency transient is coded mainly near the middle of these plots.

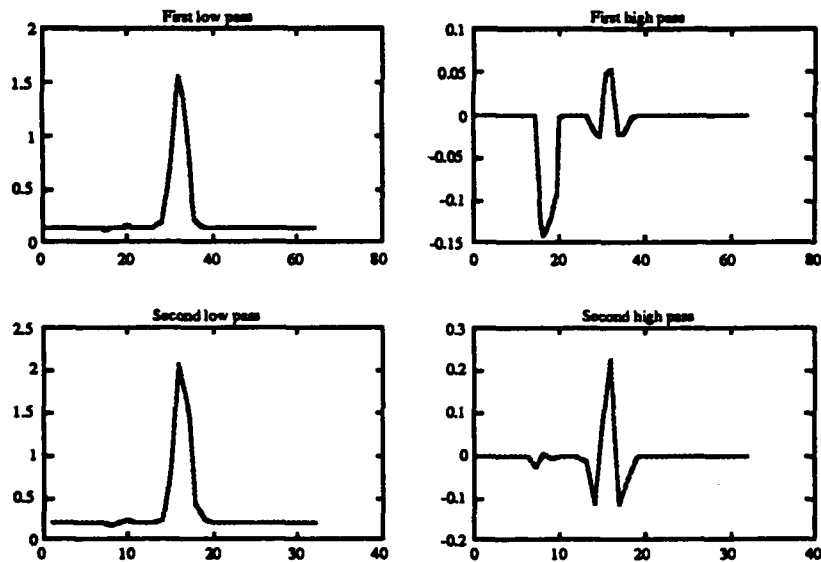
## 1-D WT Signal, $R=1$



## 1-D WT Signal, $R=1$

Similarly, the Wavelet transform (WT) reconstructs the signal exactly without compression.

## 1-D WT Decomposition, $R=1$

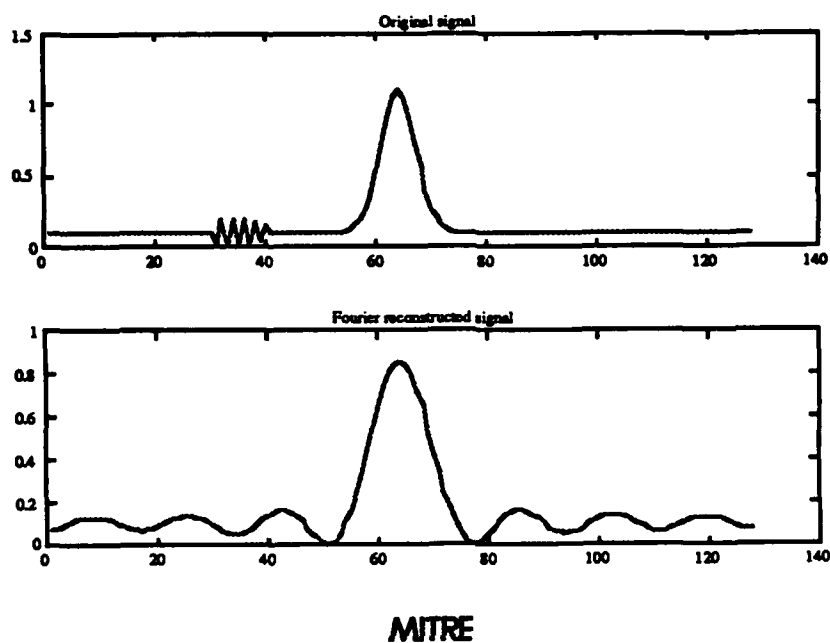


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## 1-D WT Decomposition, $R=1$

Here, the first two lowpass and highpass components are depicted (out of 5 total computed). Note that unlike the FFT, the character of the transient and Gaussian pulse are partly reflected in these frames. In particular, the time-domain on which the transient occurs is clearly visible in the first highpass component. In the following WT slides, attention should be focused on that region. Note that the lowpass filter acts as a local integrator, which is why the transient is not detected in that component; the highpass filter is a differentiator, which reacts strongly to the transient.

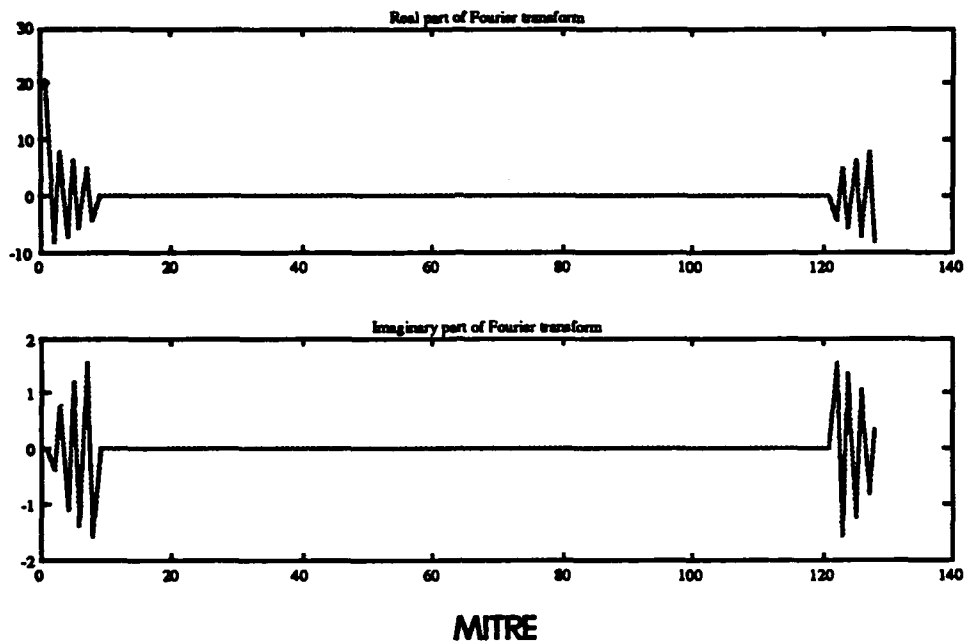
## 1-D FT Signal, R=10



1-D FT Signal, R=10

Here 10:1 compression has been applied in the Fourier domain before inversion. All traces of the transient are lost; the resulting signal is smooth but significantly deformed (in a mean-square error sense).

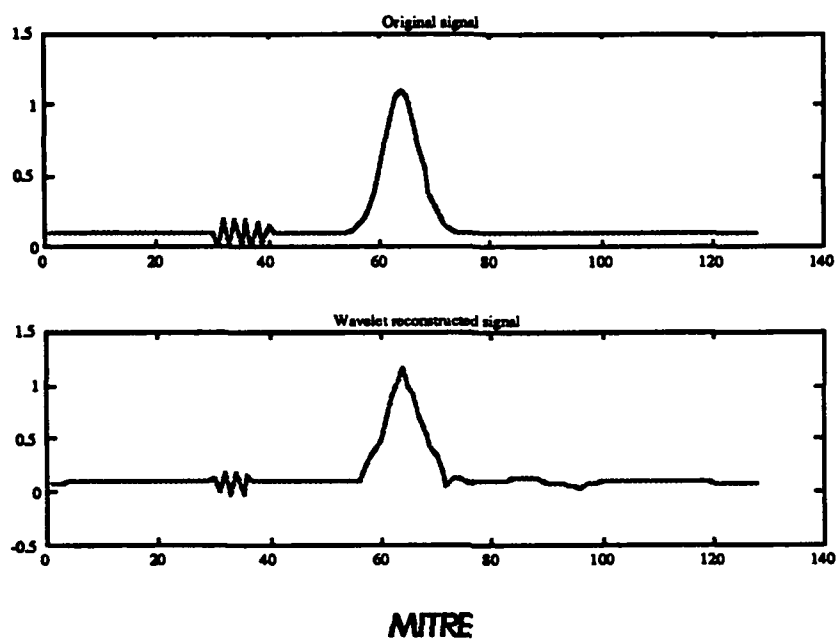
## 1-D FT Decomposition, R=10



### 1-D FT Decomposition, R=10

Compression was implemented by setting the lowest 90% of the terms (in magnitude) to 0. This had the effect of deleting all the high-frequency coefficients (which would normally reside in the center of these plots due to the cyclic nature of the FFT).

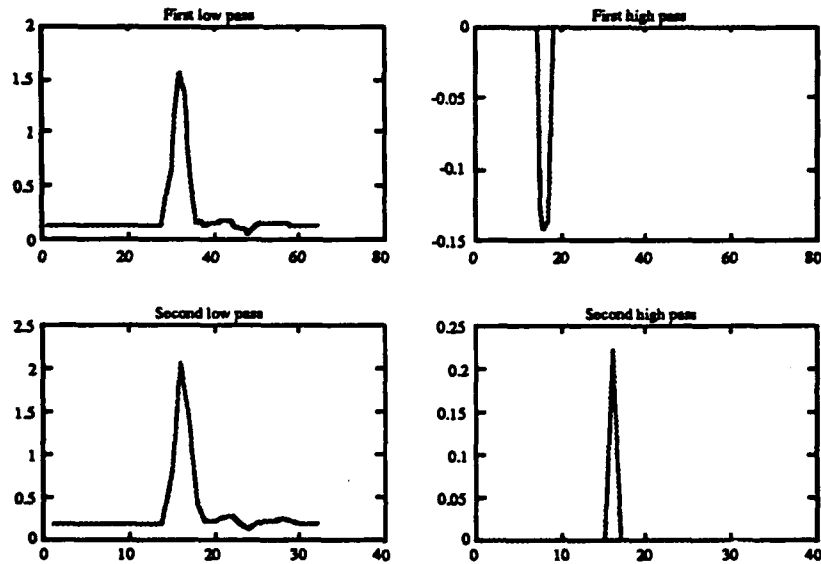
## 1-D WT Signal, R=10



## 1-D WT Signal, R=10

By contrast, the wavelet transform retains some features of the transient under 10:1 compression. In general, the signal is closer to the original in a mean-square sense than the Fourier reconstructed signal.

## 1-D WT Decomposition, R=10

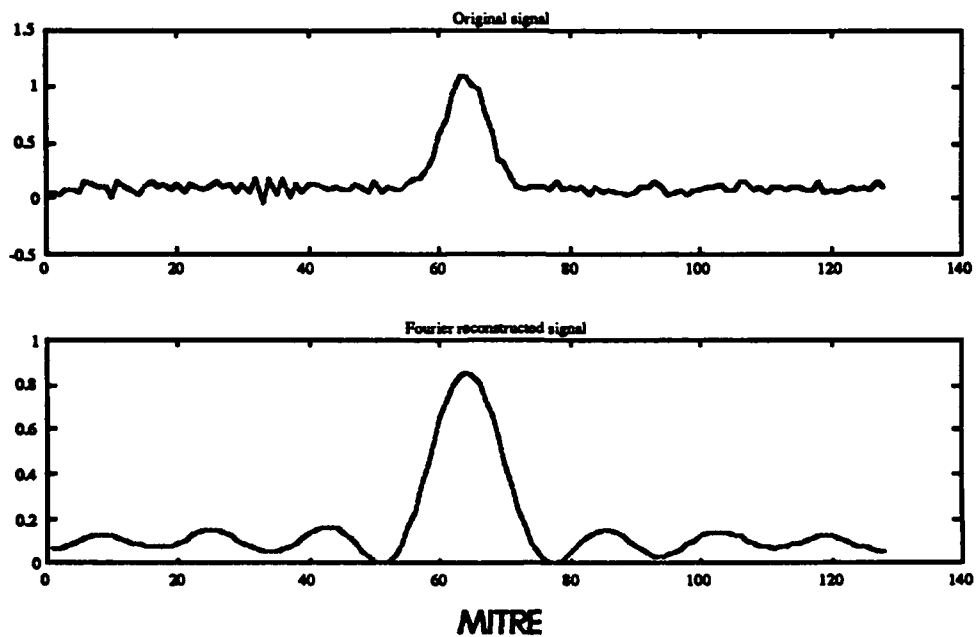


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### 1-D WT Decomposition, R=10

In the wavelet decomposition, the 10:1 compression still retains the effect of the transient in the first highpass component. Thus, a successful detector for this can be designed based on the first highpass component by setting a threshold. Again, the time of occurrence of the transient is directly available, as well as a certain measure of its intensity. We note that other types of transients may be effectively represented in other components of the wavelet decomposition, depending on their nature.

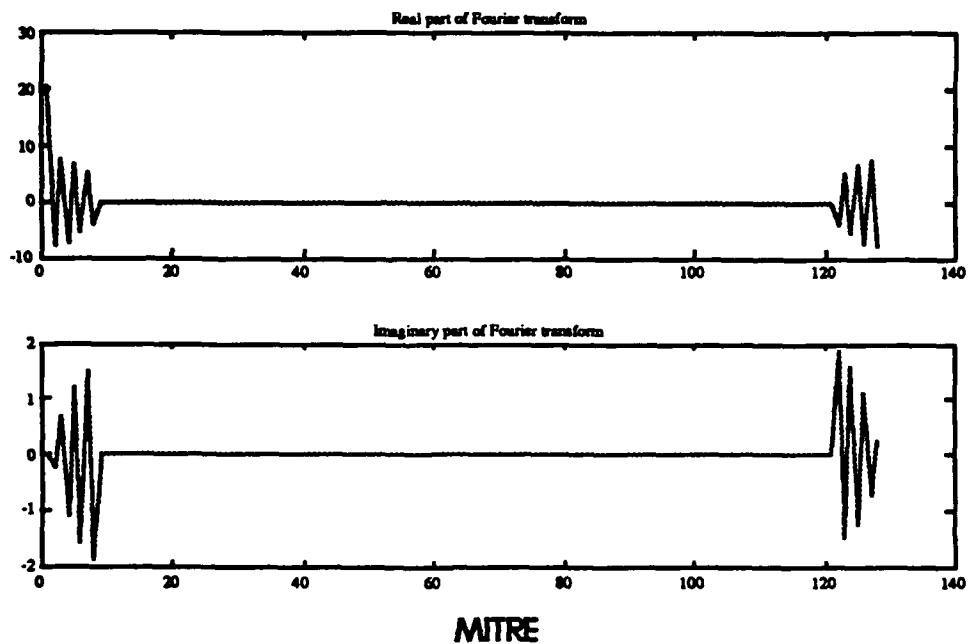
## 1-D FT Signal, -3dB N, R=10



1-D FT Signal, -3dB N, R=10

We continue the experiment by adding -3dB Gaussian noise, which has no effect in the FFT case.

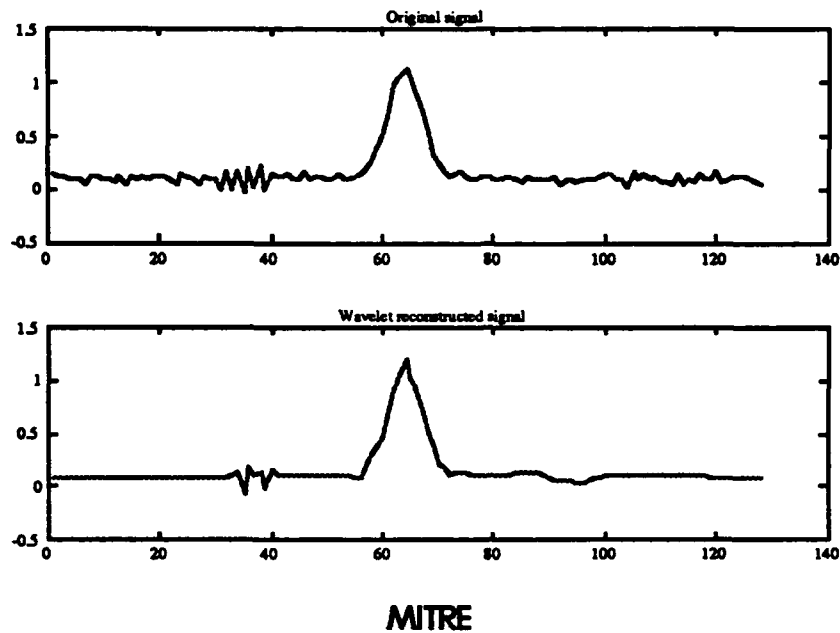
## 1-D FT Decomposition, -3dB N, R=10



1-D FT Decomposition, -3dB N, R=10

The noise is filtered away along with the transient under compression in the Fourier domain.

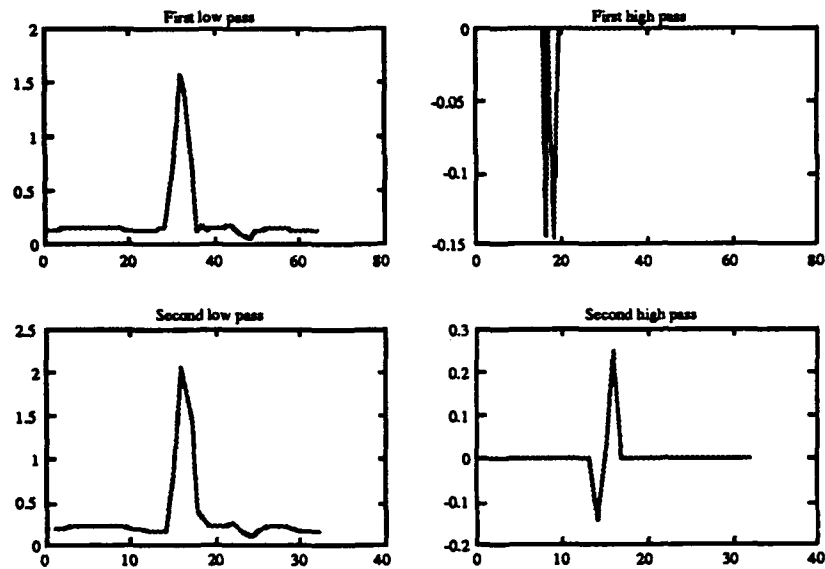
## 1-D WT Signal, -3dB N, R=10



1-D WT Signal, -3dB N, R=10

By contrast, the WT continues to retain features of the transient, while filtering most of the noise.

## 1-D WT Decomposition, -3dB N, R=10



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## 1-D WT Decomposition, -3dB N, R=10

Again, a threshold-based transient detector, using the first highpass component, would be successful here.

### Lena Original

We now consider image compression, which is perhaps the most promising application of wavelets to date. Here an original 512x512 pixel 8-bit grayscale image ("Lena") is presented. This image is an industry standard in image compression. It contains a variety of textures, from the smooth skin to the ruffled feathers in the hat. "Lena" and "House" are public domain images, obtained from a database at the University of Southern California.

LENA ORIGINAL



### Lena JPEG 44:1

This slide presents a highly compressed version of the Lena image, using an algorithm that is now the international standard for still image compression—the Joint Photographic Experts Group (JPEG) algorithm. We used public domain JPEG software courtesy of the Independent JPEG Group. This algorithm divides the image into 8x8 pixel blocks, computes a Discrete Cosine Transform (DCT) on each block, and then requantizes as necessary to achieve the desired compression ratio. Huffman coding is also used to achieve the final entropy coding of the quantized blocks. At 44:1 compression, the image displays considerable tiling as many 8x8 blocks are reduced to a single grayscale. Note that the texture on the face, shoulders, and woodwork in the background is lost.

Note that blocking of the image data is necessary for DCT-based algorithms to (a) preserve locality of image data under the transform, and (b) bound the computational complexity, which is  $O(N \log N)$ —blocking bounds the  $\log N$  factor.

LENA JPEG 44:1 2/28/93



### Lena Wavelet 45:1

At 45:1, the Wavelet Transform (WT) retains more of the structure of the original, including edge as well as texture information. In general, the WT tends to retain hard edges best, and will sacrifice textural information at high compression. Note that blocking of the image data is completely unnecessary in this case, since the WT preserves locality and has  $O(N)$  complexity. We used public domain wavelet software courtesy of the MIT Media Laboratory.

LENA WAVELET 45:1 2/28/93



**Lena JPEG 76:1 (Maximum)**

**At 76:1, the maximum compression possible on this image under JPEG, the image is unrecognizable. In fact, it could pose as excellent cubist art.**

LENA JPEG 76:1 (MAXIMUM) 2/28/93



### **Lena Wavelet 85:1**

**By contrast, at the extremely high compression of 85:1, the image is still recognizable. The eyes are still well-preserved, which is important for recognition. Among the artifacts created by the WT are aliasing effects around edges.**

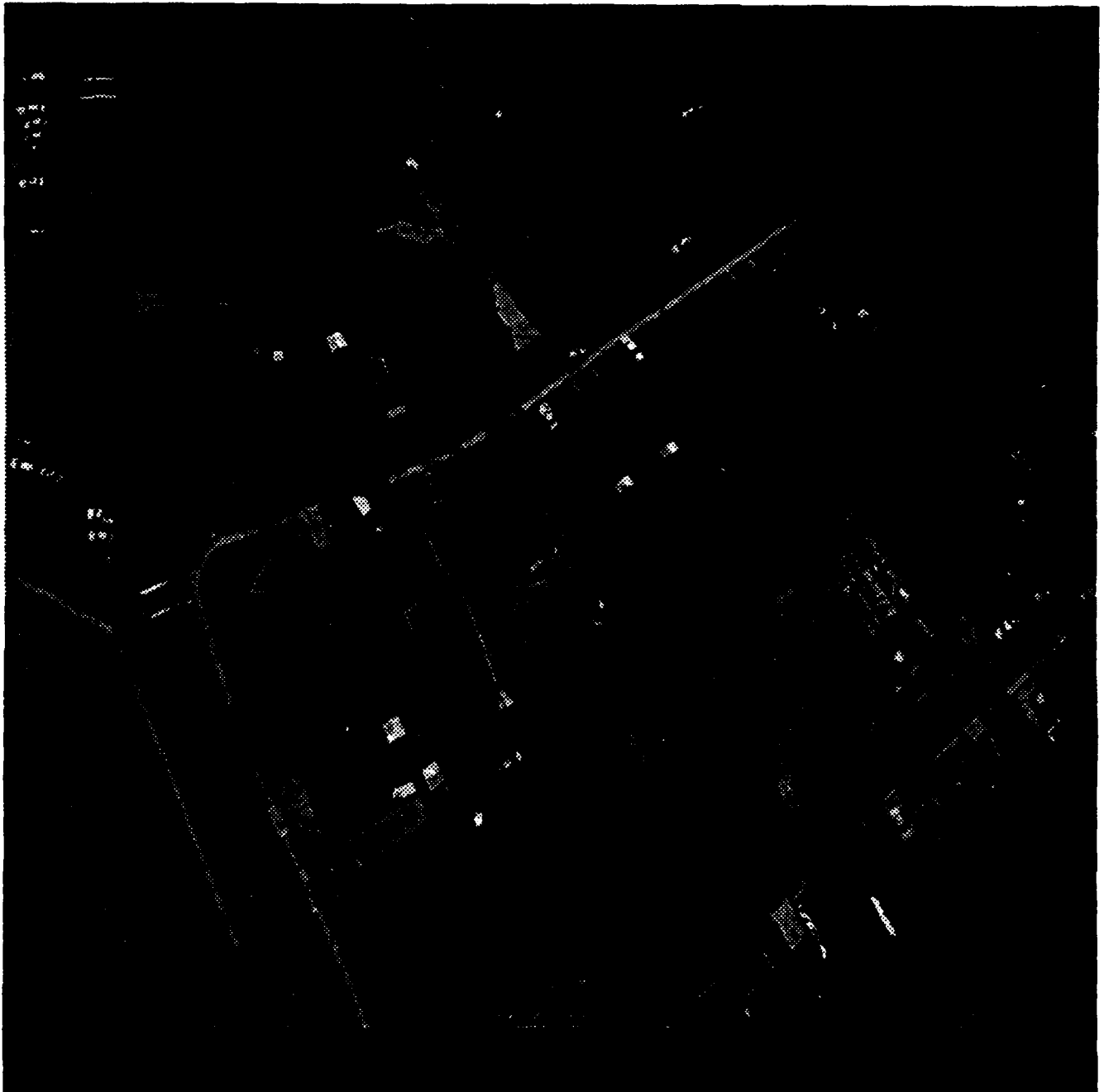
LENA WAVELET 85:1 2/28/93



### House Original

As a second example, we treat an aerial view of a residential area. (Unfortunately, this reproduction is very dark.)

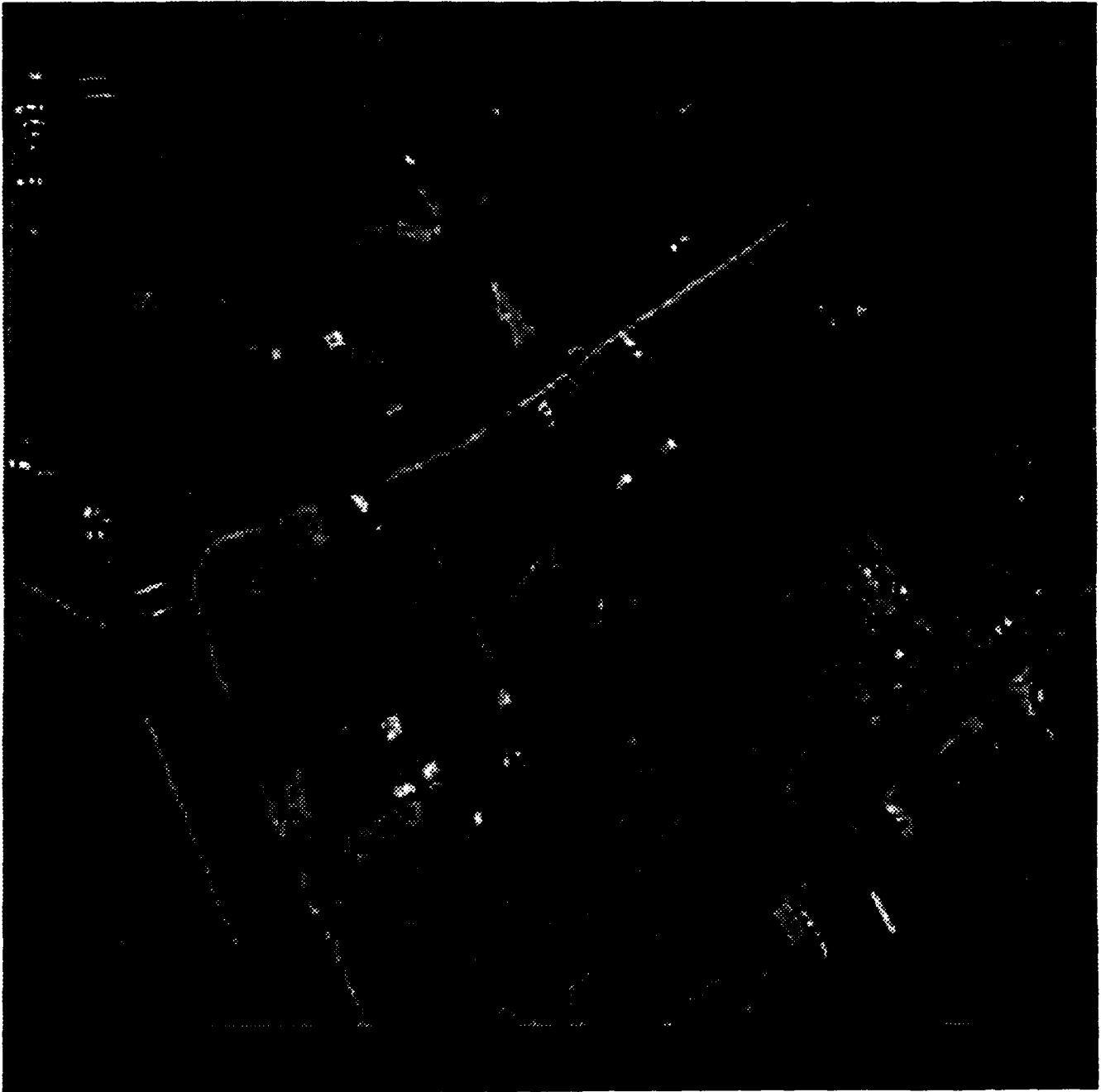
HOUSE ORIGINAL 2/28/93



### **House JPEG 40:1**

**At 40:1, JPEG produces a low-quality blocky image. The sidewalks are still visible, but the roads, trees, and many houses are obscured and tiled. The tiling is especially awkward for automatic target recognition purposes, since it introduces artificial straight lines.**

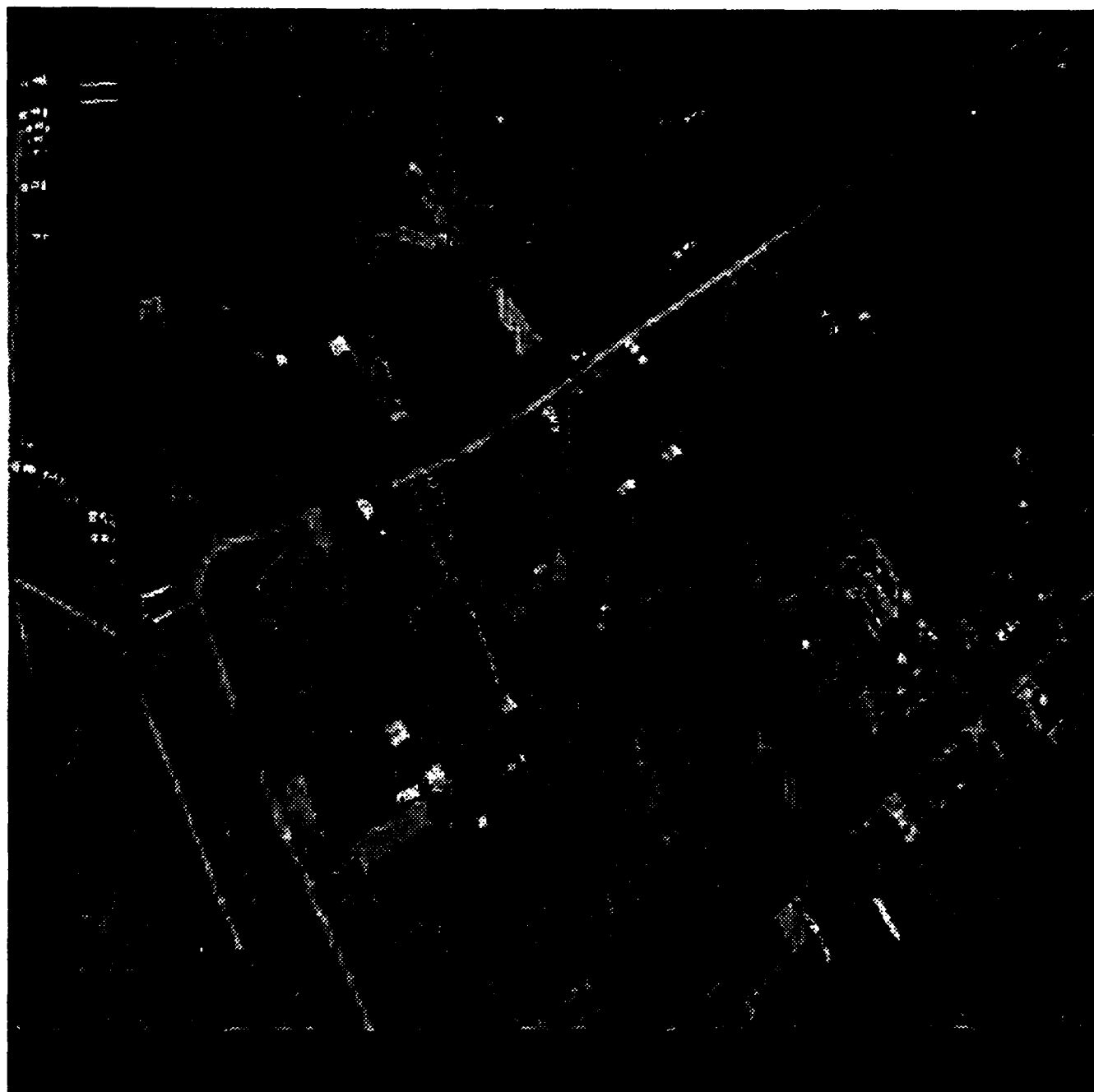
HOUSE JPEG 40:1 2/28/93



### **House Wavelet 39:1**

**At 39:1, the wavelet image is also of poor quality, indicating that the advantages of wavelet compression over JPEG are not uniform over all image types. Again, the sidewalks are better preserved than the streets, trees. Straight edges continue to be associated to houses, though.**

HOUSE WAVELET 39:1 2/28/93



### Wavelet Code

We conclude with some simple MATLAB code for wavelet processing of 1-D signals, including compression. This code uses the Haar wavelet, the simplest of the wavelet filters. We have more sophisticated code, including C codes for high performance wavelet signal processing, for one and two-dimensional signals.

```

% Pankaj Topiwala, The MITRE Corp., Burlington Rd., Bedford, MA 01730
% This program computes the wavelet transform with the Haar wavelet
% up to level 5; compression ratio R. Load data file 'data'.
% Data file must have length power of 2.
clear;
clg;

% set compression ratio...
R=1;

% retrieve (or create) data...
load data;
x=data';
% practice data: t=1:128; x= exp(- (abs(t-64).^2)/20) +.1;x=x';
M=length(x);
L=round(M/2);
alpha=1/sqrt(2);
h1=zeros(L,M);
g1=zeros(L,M);

% form the first filter matrices...
for i=1:L,
h1(i,2*(i-1)+1)=alpha;
g1(i,2*(i-1)+1)=alpha;
h1(i,2*i)=alpha;
g1(i,2*i)=-alpha;
end

% form the remaining filter matrices to level 5...
h2=h1(1:L/2,1:L);
g2=g1(1:L/2,1:L);
h3=h2(1:L/4,1:L/2);
g3=g2(1:L/4,1:L/2);
h4=h3(1:L/8,1:L/4);
g4=g3(1:L/8,1:L/4);
h5=h4(1:L/16,1:L/8);
g5=g4(1:L/16,1:L/8);

% do the orthonormal wavelet decomposition to order 5 ...
c1=h1*x;
d1=g1*x;
c2=h2*c1;
d2=g2*c1;
c3=h3*c2;
d3=g3*c2;

```

```

c4=h4*c3;
d4=g4*c3;
c5=h5*c4;
d5=g5*c4;

% plot the first two low and high pass versions...
subplot(221),plot(c1);title('First low pass');
subplot(222),plot(d1);title('First high pass');
subplot(223),plot(c2);title('Second low pass');
subplot(224),plot(d2);title('Second high pass');
pause
clg;
% PRESS THE SPACE BAR TO CONTINUE...

% threshold compression routine...
y=[d1;d2;d3;d4;d5;c5];
z=abs(y);
z=sort(z);
P=round(M*(1-1/R));if (R==1),P=1;end
val=z(P);
res=zeros(M,1);
for j=1:M,
if abs(y(j))>=val, res(j)=y(j); else res(j)=0; end
end

% reconstruction...
d1=res(1:L);
d2=res(L+1:L+L/2);
d3=res(L+L/2+1:7*L/4);
d4=res(7*L/4+1:15*L/8);
d5=res(15*L/8+1:31*L/16);
c5=res(31*L/16+1:M);

c4=h5'*c5 + g5'*d5;
c3=h4'*c4 + g4'*d4;
c2=h3'*c3 + g3'*d3;
c1=h2'*c2 + g2'*d2;
xnew=h1'*c1 + g1'*d1;

% plot the original against the reconstructed signal...
subplot(211), plot(x);title('Original signal');
subplot(212), plot(xnew);
title('Reconstructed signal');

```

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